Extra notes: Model Extensions ①

Markov-Property for an adapted measurable process (X_t):

$\forall\ t \geq s \geq 0$ and f-bounded measurable

$$E(f(X_t)/\mathcal{F}_s) = E(f(X_t)/\sigma(X_s)).$$

Consider, e.g.

$$dX_t = \beta(t, X_t)dt + \gamma(t, X_t)\, dW_t \qquad (*)$$

given functions ("drift" and "diffusion")

s.t. for a constant C

$$|\beta(t,x) - \beta(t,y)| + |\gamma(t,x) - \gamma(t,y)| \leq C|x-y|$$

$$|\beta(t,x)| + |\gamma(t,x)| \leq C(1+|x|)$$

$\forall\ t \geq 0,\ x, y \in \mathbb{R}$

Problem: Find a process (X_T) for $T \geq t \geq 0$ such that

$$X_t = x, \quad x \in \mathbb{R},$$
$$X_T = X_t + \int_t^T \beta(s, X_s)\, ds + \int_t^T \gamma(s, X_s)\, dW_s$$

For h - Borel measurable.

Denote

$$g(t,x) = E^{t,x}(h(X_T))$$

where (X_T) is the solution to (*) with initial condition $X_t = x$

(it is assumed that $E^{t,x}(h(X_T)) < \infty \ \forall x, t$)

$g(\cdot,\cdot)$ is a Borel-measurable deterministic function.

If we do not have an explicite expression X_T, we could compute $g(t,x)$ numerically by beginning at $X_t = x$ and simulating the SDE.

One way (convergence cf. Asmussen): Euler method, a particular type of Monte Carlo method:
Choose a small positive step size δ, and then set

$$X_{t+\delta} = x + \beta(t,x)\delta + \gamma(t,x)\sqrt{\delta}\,\varepsilon_1,$$

where $\varepsilon_1 \sim N(0,1)$. Then

$$X_{t+2\delta} = X_{t+\delta} + \beta(t+\delta, X_{t+\delta})\delta + \gamma(t+\delta, X_{t+\delta})\sqrt{\delta}\,\varepsilon_2$$

$\varepsilon_2 \sim N(0,1)$ $\varepsilon_1 \perp\!\!\!\perp \varepsilon_2$ etc.;

assuming $\frac{T-t}{\delta} \in \mathbb{N}$ we can find the value for X_T.

This gives a realisation of X_T, corresponding to one ω.
Now repeat this process many times and compute the average of $h(X_T)$ over all these simulations to get an approximate value for $g(t,x)$.

Note that if one would begin with a different time t and initial value for x, one would get a different answer (i.e. the "answer" is a function of t and x; this is emphasized by the notation $E^{t,x}$).

* For a concrete proper set of assumptions we refere to D. Duffie, Security Markets, AP, 1988, p. 226

Th.* Let $(X_t)_{t\geq 0}$ be a solution of (*) with initial condition given at time 0. Then, for $0 \leq t \leq T$

$$E(h(X_T)|\mathcal{F}_t) = g(t,X_t).$$

Remark: Details not easy

Problem: The Euler method for approximating this function converges slowly and gives the function value only for a pair (t,x).

Hence, translate the problem into a PDE.

Th. (Feynman-Kac)*

* For a concrete proper set of assumptions we refere to D. Duffie, Security Markets, AP, 1988, p. 226.

Consider again the SDE (*) i.e.

$$dX_t = \beta(t,X_t)dt + \gamma(t,X_t)dW_t .$$

Let $h(x)$ be a Borel-measurable function. Fix $T > 0$ and let $t \in [0,T]$ be given.

Define the function

$$g(t,x) = E^{t,x}(h(X_T)).$$ Assume $E^{t,x}|h(X_T)| < \infty$ for all t and x. Then $g(t,x)$ satisfies the PDE

$$g_t(t,x) + \beta(t,x)g_x(t,x) + \frac{1}{2}\gamma^2(t,x)g_{xx}(t,x) = 0 \quad (**)$$

and the terminal condition

$$g(T,x) = h(x) \text{ for all } x.$$

Numerical algorithms for solving (**) converge quickly in the case of one-dimensional x considered here and give the function $g(t,x)$ for all values of (t,x) simultaneously.

* For a concrete proper set of assumptions we again refere to D. Duffie, p. 226

Th. (Discounted Feynman-Kac)*

Consider again the SDE

$$dX_t = \beta(t,X_t)\,dt + \gamma(t,X_t)\,dW_t$$

Let $h(x)$ be a Borel-measurable function and r be constant. Fix $T>0$ and let $t \in [0,T]$. Define the function

$$f(t,x) = E^{t,x}\left(e^{-r(T-t)} h(X_T)\right)$$

and assume $E^{t,x}(h(X_T)) < \infty$ for all t,x.

Then $f(t,x)$ satisfies the PDE

$$f_t(t,x) + \beta(t,x) f_x(t,x) + \tfrac{1}{2}\gamma^2(t,x) f_{xx}(t,x) = r f(t,x)$$

and the terminal condition

$$f(T,x) = h(x) \quad \text{for all } x.$$

(we can efficiently compute disc. cond. expect.)

Extra notes on SV and short rates

assume that $S = (S_t)_{t \ge 0}$ satisfies

$$dS_t = \mu S_t\,dt + \sigma(t) S_t\,dB_t \qquad (*)$$

with

$$\sigma(t) = e^{Y_t}, \quad Y_t \text{ s.t.}$$

$$dY_t = \overbrace{(\alpha - \beta Y_t)}^{a_t}\,dt + \overbrace{\gamma}^{b_t}\,d\tilde{B}_t$$

as on the slides.

$$f(y) = e^{y} = \frac{\partial f}{\partial y}(y) = \frac{\partial^2 f}{\partial y^2}(y)$$

$$\Downarrow \text{ Itô}$$

$$d\sigma(t) = \Big(\underbrace{(\alpha - \beta Y_t)}_{a_t}\underbrace{\sigma(t)}_{\frac{\partial f}{\partial y}(Y_t)} + 0 + \frac{1}{2}\overbrace{\gamma^2}^{b_t^2}\underbrace{\sigma(t)}_{\frac{\partial^2 f}{\partial y^2}(Y_t)}\Big)dt + \overbrace{\gamma}^{b_t}\underbrace{\sigma(t)}_{\frac{\partial f}{\partial y}(Y_t)}\,d\tilde{B}_t$$

$$= \sigma(t)\left(\alpha - \beta Y_t + \frac{1}{2}\gamma^2\right)dt + \sigma(t)\gamma\,d\tilde{B}_t \qquad (**)$$

$$\Downarrow$$

$$d\langle S, \sigma \rangle_t = \sigma(t)^2 S_t \gamma\, d\langle B, \tilde{B} \rangle_t$$

$$dS_t = \dots + \sigma(t) S_t\,dB_t$$
$$d\sigma(t) = \dots + \sigma(t)\gamma\,d\tilde{B}_t$$

With $d\langle B, \tilde{B}\rangle_t = \varsigma dt$, we have

$d\langle S, \sigma\rangle_t = \sigma^2(t) S_t \gamma \varsigma dt$

Hence, $\varsigma < 0$ is needed in order to ensure negative covariation between prices and volatility.

Other application of the OU-Process:

$dr_t = (\alpha - \beta r_t)dt + \sigma dW_t$ - (W_t) sBM

$S_t^0 = B_t = e^{\int_0^t r_s ds}$ - Vašiček-model

Or, more generally for k, θ, σ - bounded Borel functions

$dr_t = k_t(\theta_t - r_t)dt + \sigma_t dW_t$ (*)

~ Hull-White model (k-positive function)

Cf. Method of Variation of Constants:

The solution is (r_0 - given)

$$r_t = e^{-K_t}\left(r_0 + \int_0^t e^{K_s} k_s \theta_s\, ds + \int_0^t e^{K_s}\sigma_s\, dW_s\right),$$

$K_t = \int_0^t k_s\, ds$. Remark: (r_t) - Gaussian process.

(r_t) is called to be the short rate

$$r_t = F(t, X_t) = e^{-K_t} X_t, \quad X_t = r_0 + \int_0^t e^{K_s} k_s \theta_s\, ds + \int_0^t e^{K_s}\sigma_s\, dW_s$$

i.e. (notation)

$$dX_t = \underbrace{e^{K_t} k_t \theta_t}_{a_t}\, dt + \underbrace{e^{K_t}\sigma_t}_{b_t}\, dW_t$$

$$F(t,x) = e^{-K_t} x \Rightarrow \frac{\partial F}{\partial x}(t,x) = e^{-K_t},$$

$$\frac{\partial^2 F}{\partial x^2}(t,x) = 0, \quad \frac{\partial F}{\partial t}(t,x) = -k_t e^{-K_t} x$$

$$\Rightarrow \frac{\partial F}{\partial t}(t, X_t) = -k_t e^{-K_t} X_t$$

$\Downarrow$ Itô

$$dr_t = \Big(\underbrace{e^{-\mu_t}}_{\frac{\partial F}{\partial x}(t,X_t)} \underbrace{e^{\mu_t} k_t \theta_t}_{a_t} - \underbrace{k_t \overbrace{e^{-\mu_t} X_t}^{r_t}}_{\frac{\partial F}{\partial t}(t,X_t)} + 0\Big)dt$$

$$+ \underbrace{e^{-\mu_t}}_{\frac{\partial F}{\partial x}(t,X_t)} \underbrace{e^{\mu_t} \sigma_t}_{b_t} dW_t$$

$$= k_t(\theta_t - r_t)dt + \sigma_t dW_t$$

CIR-process is also used for short rate modelling

$$dr_t = (\alpha - \beta r_t)dt + \sigma\sqrt{r_t}\, dW_t; \quad r_0 > 0 \text{ given.}$$

The solution can be approximated by Monte Carlo simulations and many of its properties can be determined even though we do not have an explicit formula for it.

The choice of the CIR process as the driving process for the instantaneous volatility leads to closed form expressions for characteristic functions of various quantities of interest in this model, enabeling e.g. the use of Fourier transform methods for option pricing.

Valuation with stochastic (r_t)

Let X be a T-claim

$B_t = S_t^0 = \exp\left(\int_0^t r_s \, ds\right)$ with

$E_Q\left(\frac{|X|}{S_T^0}\right) < \infty$.

Most important example:

$X = 1$ - zerobond.

Price

$$P(t,T) = S_t^0 E_Q\left(\frac{1}{S_T^0} \Big| \mathcal{F}_t\right)$$

$$= E_Q\left(\underbrace{e^{-\int_t^T r_s\,ds}}_{\text{one random variable}} \Big| \mathcal{F}_t\right)$$

For X - more complex

$$P_t = S_t^0 E_Q\left(\frac{X}{S_T^0} \Big| \mathcal{F}_t\right)$$

two random variables

$$\frac{dQ^T}{dQ} = \frac{P(T,T)}{P(0,T) S_T^0}$$

- assumed to be Q-integrable.

We further assume that

$$\left(\frac{1}{P(0,T)} \cdot \frac{P(t,T)}{S_t^0}\right)_t$$ - is a true martingale w.r.t. Q.

For $t \leq T$

$$Z_t = \frac{dQ^T}{dQ}\Big|_{\mathcal{F}_t} = E_Q\left(\frac{dQ^T}{dQ}\Big|\mathcal{F}_t\right)$$

$\left(E_Q\left(\frac{dQ^T}{dQ}\Big|\mathcal{F}_t\right)\right)$ - martingale w.r.t. Q by tower property

assumption (bond mark.)

$$= \frac{P(t,T)}{P(0,T)\, S_t^0} \qquad t \leq T$$

Thus,

" $\frac{dQ^T}{dQ^T} dQ$ "

$$\underbrace{P_t}_{} = S_t^0 E_Q\left(\frac{X}{S_T^0}\Big|\mathcal{F}_t\right) = S_t^0 E_{Q^T}\left(\frac{X}{S_T^0}\frac{Z_t}{Z_T}\Big|\mathcal{F}_t\right)$$

measurable

$$= S_t^0 E_{Q^T}\left(\frac{X}{S_T^0}\underbrace{\frac{P(t,T)}{P(0,T)\,S_t^0}}_{Z_t}\cdot\underbrace{\frac{P(0,T)\,S_T^0}{P(T,T)}}_{= \frac{1}{1/Z_T}}\Big|\mathcal{F}_t\right)$$

$$= \underline{P(t,T)\, E_{Q^T}(X|\mathcal{F}_t)}$$

two times one random variable

Note: With the measure also the distribution of a r.v./process changes

Extra remarks on Lévy processes

0) Consider càdlàg processes, $L_0 = 0$.

1) Independent increments: $X \in PII$,
if $X_t - X_s \perp\!\!\!\perp \mathcal{F}_s \quad \forall\, 0 \le s < t < \infty$.

⚠ $X \in PII \not\Rightarrow X$ - semimartingale

but

$$X = D + S$$ — semimartingale

deterministic process (maybe of infinite variation)

2) Stationary increments:
The distribution of $X_t - X_s$ depends only on $t-s$ and therefore

$$\mathcal{L}aw(X_{t+h} - X_{s+h}) = \mathcal{L}aw(X_t - X_s) \quad \forall\, h > 0$$

3) $\forall\, \varepsilon > 0 \quad \lim_{s \to t} P(|L_s - L_t| > \varepsilon) = 0$

(i.e. continuous in probability)

Note (L_t) - Lévy, $S_t = S_0 e^{L_t}$, and 15

assume $E_Q(|f(S_T)|) < \infty$, then

$$e^{-r(T-t)} E_Q(f(S_0 e^{L_T}) | \mathcal{F}_t)$$

$$= e^{-r(T-t)} E_Q(f(\underbrace{S_0 e^{L_t}}_{S_t} \underbrace{e^{L_T - L_t}}_{\text{independent}}) | \mathcal{F}_t)$$

measurable

$$= \varphi(S_t), \text{ where (Prop)}$$

$$\varphi(s) = e^{-r(T-t)} E_Q(f(s e^{L_T - L_t})), \ \forall s > 0.$$

Hence, a framework with "good valuation properties".

Decompositions

$(X_t)_{t\geq 0}$ - semimartingale with $X_0 = 0$

$\underbrace{X_t - \sum_{s\leq t} \Delta X_s \mathbb{1}_{\{|\Delta X_s|>1\}}}$ - process with bounded jumps

special semimartingale.
Hence, unique decomposition into a local martingale and a predictable process of bounded variation $A + M$

$$\Downarrow$$

$$X_t = \underbrace{A_t}_{\text{depends on the truncation}} + M_t + \underbrace{\sum_{s\leq t} \Delta X_s \mathbb{1}_{\{|\Delta X_s|>1\}}}_{\text{depends on the truncation}}$$

(truncation function here:
$h(x) = x\mathbb{1}_{\{|x|\leq 1\}} \Rightarrow (x-h(x)) = x\mathbb{1}_{\{|x|>1\}}$)

$M_t = \underbrace{M_t^c}_{M^c\text{-continuous}} + M_t^d$

M^d - purely discontinuous, i.e. M^dN - local martingale $\forall N$ continuous local martingales

Remark: It turns out that the continuous martingale component M^c of a semimartingale does not depend on the function h.

Lévy - case:

$A_t = bt$ and $M^c_t = \sqrt{c}\, W_t$ (W_t)-sBM

μ - random measure of jumps of

$L = (L_t)_{t \geq 0}$ - Lévy - process.

$\mu(\omega; (0,t] \times A)$ - counts how many jumps of size within A occure for path ω from 0 to t.

$$\nu((0,t] \times A) = t\underbrace{F(A)}_{\text{Lévy-measure, i.e.}} = E(\mu(\cdot\,; (0,t] \times A))$$

$F(A) = E(\#\{t \in (0,1] : \Delta L_t \neq 0, \Delta L_t \in A\})$ counts the expected number per unite time of jumps whose size belongs to A. Does not depend on the function h.

M^d - purely discontinuous (local) martingale

$$M_t^d = \int_0^t \int_{\mathbb{R}} x \mathbb{1}_{\{|x| \le 1\}} (\overbrace{\mu(ds, dx)}^{\mu} - \overbrace{ds\, F(dx)}^{\nu})$$

Then

$$L_t = \underbrace{bt}_{A_t} + \underbrace{\sqrt{c}\, W_t}_{M_t^c} + \underbrace{\int_0^t \int_{\mathbb{R}} x \mathbb{1}_{\{|x| \le 1\}} (\mu(ds, dx) - ds\, F(dx))}_{M_t^d}$$

$$+ \underbrace{\int_0^t \int_{\mathbb{R}} x \mathbb{1}_{\{|x| > 1\}} \mu(ds, dx)}_{\text{"big jumps"}}$$

- The four terms in the decomposition are independent
- $F(dx)$ and c do not depend on the truncation
- $M_t^d = \lim_{\varepsilon \to 0} \left(\sum_{s \le t} \Delta L_s \mathbb{1}_{\{\varepsilon \le |\Delta L_s| \le 1\}} - t \int x \mathbb{1}_{\{\varepsilon \le |x| \le 1\}} F(dx) \right)$

  (convergence is a.s. and uniform in t on $[0, T]$)
- $\int_0^t \int_{\mathbb{R}} x \mathbb{1}_{\{|x| > 1\}} \mu(ds, dx) = \sum_{s \le t} \Delta L_s \mathbb{1}_{\{|\Delta L_s| > 1\}}$

Subordination

- (T_t) - subordinator with Lévy triplet $(\tilde{b}, 0, \rho)$.
- The moment generating function of (T_t) is $E(e^{uT_t}) = e^{tl(u)}$ $\forall u$ such that the moment generating function exists, where

$$l(u) = \tilde{b}u + \int_0^\infty (e^{ux} - 1)\rho(dx)$$

$$\rho((-\infty, 0]) = 0, \quad \int_0^\infty (x \wedge 1)\rho(dx) < \infty, \quad \tilde{b} \geq 0.$$

- (X_θ) - Lévy process with characteristic function

$$E(e^{iuX_\theta}) = e^{\theta \psi(u)}, \qquad \forall u \in \mathbb{R}$$

$$\psi(u) = iub - \frac{u^2}{2}c + \int (e^{iux} - 1 - iuh(x))F(dx)$$

and Lévy triplet (b, c, F).

- Define (Y_t) by $Y_t(\omega) = X(T_t(\omega), \omega)$.
- Thus, for X, T - independent

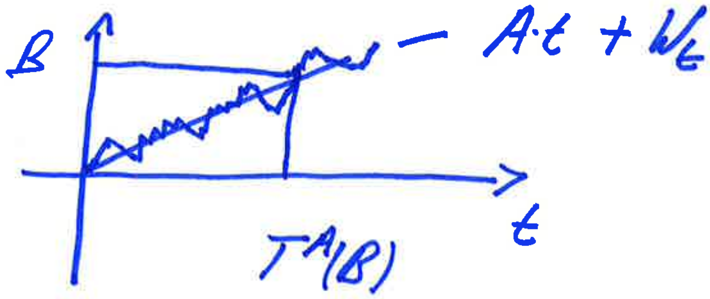
$$\Downarrow$$

$$E(e^{iuY_t}) = e^{tl(\psi(u))} \quad \text{since}$$

$$\underline{E(e^{iuY_t})} = E(e^{iuX(T_t)}) = E(\overbrace{E(e^{iuX(T_t)} \mid \mathcal{F}^T_\infty)}^{\text{characteristic function of } X(T_t) \text{ given } T_t \text{ at } u})$$

$$= \underbrace{E(e^{T_t \psi(u)})}_{\text{moment generating function of } T_t \text{ at } \psi(u)} = \underline{e^{tl(\psi(u))}}$$

Inverse Gaussian



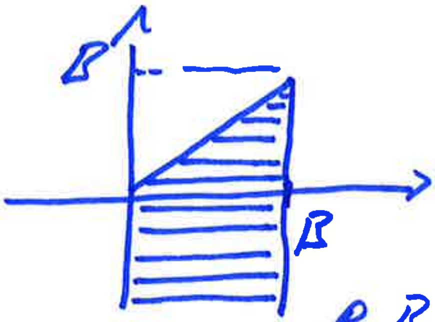
$$T^A(B) = \inf\{t \geq 0 : At + W_t \geq B\}$$

Recall W7, Ex. 3

$$\left.\begin{array}{l} \hat{W} = (\alpha t + W_t)_{t \in [0,T]} \\ \hat{M}_T = \sup\limits_{t \in [0,T]} \hat{W}_t \\ (W_t) - \text{SBM} \end{array}\right\} \Rightarrow (\hat{M}_T, \hat{W}_T)$$

has density:

$$f(m,w) = \frac{2(2m-w)}{T\sqrt{2\pi T}} e^{\alpha w - \frac{1}{2}\alpha^2 T - \frac{1}{2T}(2m-w)^2} \mathbb{1}_{\{w \leq m\}} \mathbb{1}_{\{m > 0\}}$$



$$P(\hat{M}_T \le B) = \int_{-\infty}^{B}\int_{w^+}^{B} \frac{2(2m-w)}{T\sqrt{2\pi T}}\, e^{\alpha w - \frac{1}{2}\alpha^2 T - \frac{1}{2T}(2m-w)^2}\, dm\, dw$$

$$= \frac{1}{\sqrt{2\pi T}} \int_{-\infty}^{B} e^{\alpha w - \frac{1}{2}\alpha^2 T - \frac{1}{2T} w^2}\, dw$$

Ex. 4, W7

$$- \frac{1}{\sqrt{2\pi T}} \int_{-\infty}^{B} e^{\overset{(2)}{\alpha w} - \overset{(3)}{\frac{1}{2}\alpha^2 T} - \overset{(1)}{\frac{1}{2T}(2B-w)^2}}\, dw$$

Observe for the first integral

$$-\frac{1}{2T}(w-\alpha T)^2 = -\frac{1}{2T}w^2 + \alpha w - \frac{1}{2}\alpha^2 T,$$

and for the second integral

$$-\frac{1}{2T}((w-2B)-\alpha T)^2 = \underset{(1)}{-\frac{1}{2T}(w-2B)^2} + \underset{(2)}{\alpha w} - 2\alpha B - \underset{(3)}{\frac{1}{2}\alpha^2 T}$$

to be corrected

Thus,

$$P(\hat{M}_T \le B) = \frac{1}{\sqrt{2\pi T}} \int_{-\infty}^{B} e^{-\frac{1}{2T}(w-\alpha T)^2} dw$$

$$- \frac{e^{2\alpha B}}{\sqrt{2\pi T}} \int_{-\infty}^{B} e^{-\frac{1}{2T}(w-2B-\alpha T)^2} dw$$

$$y = \frac{w-\alpha T}{\sqrt{T}} \qquad z = \frac{w-2B-\alpha T}{\sqrt{T}}$$

$$\left|\frac{dy}{dw}\right|^{-1} = \sqrt{T} \qquad \left|\frac{dz}{dw}\right|^{-1} = \sqrt{T}$$

$$= \frac{\sqrt{T}}{\sqrt{2\pi T}} \int_{-\infty}^{\frac{B-\alpha T}{\sqrt{T}}} e^{-\frac{1}{2}y^2} dy - \frac{e^{2\alpha B}\sqrt{T}}{\sqrt{2\pi T}} \int_{-\infty}^{\frac{-B-\alpha T}{\sqrt{T}}} e^{-\frac{1}{2}z^2} dz$$

$$= N\left(\frac{B-\alpha T}{\sqrt{T}}\right) - e^{2\alpha B} N\left(\frac{-B-\alpha T}{\sqrt{T}}\right)$$

$\alpha = A$ - notation

exercises ... slides

$$P(T^A(B) \le t) = P(\hat{M}_t \ge B)$$

$$= 1 - P(\hat{M}_t \le B)$$

Hence,

$$P(T^A(B) \le t) = 1 - P(\hat{M}_t \le B)$$

$$= 1 + e^{2AB} N\left(\frac{-B-At}{\sqrt{t}}\right) - N\left(\frac{B-At}{\sqrt{t}}\right)$$

Thus,

$$\frac{\partial}{\partial t} P(T^A(B) \le t)$$

$$\parallel$$

$$\frac{\partial}{\partial t}\left(1 + \frac{e^{2AB}}{\sqrt{2\pi}} \int_{-\infty}^{\frac{-B-At}{\sqrt{t}}} e^{-\frac{1}{2}y^2}\,dy - \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\frac{B-At}{\sqrt{t}}} e^{-\frac{1}{2}y^2}\,dy\right)$$

$$\parallel$$

$$\frac{e^{2AB}\, e^{-\frac{1}{2}\frac{(B+At)^2}{t}}}{\sqrt{2\pi}}\left(-\frac{A}{\sqrt{t}} - \frac{1}{2}\frac{(-B-At)}{t^{3/2}}\right)$$

$$- \frac{e^{-\frac{1}{2}\frac{(B-At)^2}{t}}}{\sqrt{2\pi}}\left(-\frac{A}{\sqrt{t}} - \frac{1}{2}\frac{B-At}{t^{3/2}}\right)$$

$$\parallel$$

$$\frac{e^{-\frac{\overbrace{-4ABt + B^2 + 2ABt}^{-2ABt} + A^2t^2}{2t}}}{\sqrt{2\pi}}\left(-\cancel{\frac{A}{\sqrt{t}}} + \frac{1}{2}\cdot\frac{B}{t^{3/2}} + \cancel{\frac{At}{2t^{3/2}}}\right)$$

$$+ \frac{e^{-\frac{1}{2}\frac{(B-At)^2}{t}}}{\sqrt{2\pi}}\left(\cancel{\frac{A}{\sqrt{t}}} + \frac{1}{2}\frac{B}{t^{3/2}} - \frac{1}{2}\cancel{\frac{At}{t^{3/2}}}\right)$$

$$e^{-\frac{1}{2}(B-At)^2}$$

Hence,

$$\frac{\partial}{\partial t} P(T^A(B) \le t) = \frac{B}{\sqrt{2\pi t}\, t} e^{-\frac{1}{2}\frac{(B-At)^2}{t}}$$

$$b = B^2 \quad \overset{A\ge 0,\, B\ge 0}{\Longleftrightarrow} \quad B = \sqrt{b}$$

$$a = A^2 \quad \Longleftrightarrow \quad A = \sqrt{a}$$

$$\frac{\partial}{\partial t} P(T^A(B) \le t) = \frac{\sqrt{b}}{\sqrt{2\pi}\, t^{3/2}} e^{-\frac{(\sqrt{b}-\sqrt{a}t)^2}{2t}}$$

$$= \frac{\sqrt{b}}{\sqrt{2\pi}\, t^{3/2}} e^{-\frac{(b - 2\sqrt{ab}t + at^2)}{2t}}$$

$$= \frac{\sqrt{b}}{\sqrt{2\pi}\, t^{3/2}} e^{\sqrt{ab}} e^{-\frac{at + \frac{b}{t}}{2}}$$

$$= \frac{\sqrt{b}\, e^{\sqrt{ab}}}{\sqrt{2\pi}} t^{-3/2} e^{-\frac{(at + b/t)}{2}}$$

$$= \underbrace{\frac{\sqrt{b}\, e^{\sqrt{ab}}}{\sqrt{2\pi}}}_{c(a,b)} t^{\nu - 1} e^{-\frac{(at + b/t)}{2}} \quad \text{for } \nu = -\frac{1}{2}$$

Characteristic function NIG Distribution

Generalized Hyperbolic

25

$$E(e^{iuL}) = E(e^{iu(\mu+\beta\sigma^2+\sigma\varepsilon)})$$

$$= E(E(e^{iu(\mu+\beta\sigma^2+\sigma\varepsilon)}\,|\,\sigma))$$

σ-measurable

ε-independent

$$= E\left(\int_{\mathbb{R}} e^{iux}\frac{1}{\sqrt{2\pi\sigma^2}}\exp\left(-\frac{1}{2}\frac{(x-(\mu+\beta\sigma^2))^2}{\sigma^2}\right)dx\right)$$

rv

$$= \int_0^\infty\int_{\mathbb{R}} e^{iux}\frac{1}{\sqrt{2\pi y}}\exp\left(-\frac{1}{2}\frac{(x-(\mu+\beta y))^2}{y}\right)p_{IG}(y)\,dx\,dy$$

$$= \int_{\mathbb{R}} e^{iux}\underbrace{\int_0^\infty \frac{1}{\sqrt{2\pi y}}\exp\left(-\frac{1}{2}\frac{(x-(\mu+\beta y))^2}{y}\right)p_{IG}(y)\,dy}_{\text{Density}}\,dx$$