Extra notes FTAP I

Discrete time (finite) case

Remarks on the setup:

- Working with a finite probability space avoids some technical problems, e.g. all real-valued random variables are integrable (finite sum).

- The economical interpretation of $\mathcal{F}_n$ here is that the investors know past and present prices but quite reasonably, they do not know the future ones.

- $S_n^0 = (1+r)^n$ is also the price change factor of a riskless asset until n, and hence, $1/(1+r)^n$ is the discount factor from time n to time 0. The assets indexed by $i=1,\dots,d$ are called "risky assets".

Remark (self-financing)

$$\phi_n \cdot S_n = \phi_{n+1} \cdot S_n \quad \forall n$$

$$\Updownarrow$$

$$-\phi_n \cdot S_n = -\phi_{n+1} \cdot S_n \quad \forall n$$

$$\Updownarrow$$

$$\underbrace{\phi_{n+1} \cdot S_{n+1}}_{= V_{n+1}(\phi)} - \underbrace{\phi_n \cdot S_n}_{= V_n(\phi)} = \phi_{n+1} \cdot (S_{n+1} - S_n) \quad \forall n$$

$$\Updownarrow$$

$$V_{n+1}(\phi) = V_n(\phi) + \phi_{n+1} \cdot (\underbrace{S_{n+1} - S_n}_{= \Delta S_{n+1}}) \quad \forall n$$

- notation

Hence,

$$\phi_n \cdot S_n = \phi_{n+1} \cdot S_n \quad \forall n$$

$$\Updownarrow$$

$$V_{n+1}(\phi) \overset{(*)}{=} V_n(\phi) + \phi_{n+1} \cdot \Delta S_{n+1}$$

Remark to Proposition 1

- $\Delta \tilde{S}_j = \frac{1}{S_j^0}\cdot S_j - \frac{1}{S_{j-1}^0}\cdot S_{j-1} = \tilde{S}_j - \tilde{S}_{j-1}$

vectors

- $\Delta \tilde{S}_j^0 = \frac{S_j^0}{S_j^0} - \frac{S_{j-1}^0}{S_{j-1}^0} = 1 - 1 = 0$

Proof of Proposition 1

(I) => (II): For any $\hat{n} \in \{1, \dots, N\}$ we have

$$V_0(\phi) + \phi_1\cdot(S_1 - S_0)$$

$$\overset{(*)\text{ for } n=0}{=} \underbrace{V_1(\phi) \qquad + \phi_2\cdot(S_2 - S_1)}$$

$$\overset{(*)\text{ for } n=1}{=} V_2(\phi)$$

$$\vdots$$

$$\overset{(*)\text{ for } n=\hat{n}-2}{=} \underbrace{V_{\hat{n}-1}(\phi) \qquad + \phi_{\hat{n}}\cdot(S_{\hat{n}} - S_{\hat{n}-1})}$$

$$\overset{(*)\text{ for } n=\hat{n}-1}{=} V_{\hat{n}}(\phi)$$

Hence,

$$V_n(\phi) = V_0(\phi) + \sum_{j=1}^{n} \phi_j \cdot (S_j - S_{j-1})$$

$$= V_0(\phi) + \sum_{j=1}^{n} \phi_j \cdot \Delta S_j$$

Conversely, (II) $\Rightarrow$ (I)

$V_n(\phi) = V_0(\phi) + \sum_{j=1}^{n} \phi_j \cdot \Delta S_j$ holds $\forall n$

For a fixed $n \in \{0, \dots, N-1\}$ consider

$$\begin{aligned} V_{n+1}(\phi) &= V_0(\phi) + \sum_{j=1}^{n+1} \phi_j \cdot \Delta S_j \\ -V_n(\phi) &= -V_0(\phi) - \sum_{j=1}^{n} \phi_j \cdot \Delta S_j \end{aligned}$$

$$V_{n+1}(\phi) - V_n(\phi) = \phi_{n+1} \cdot \Delta S_{n+1}$$

$$\Updownarrow$$

$$V_{n+1}(\phi) = V_n(\phi) + \phi_{n+1} \cdot \Delta S_{n+1};$$

hence, (*) holds $\forall n$

$$\Updownarrow$$

(I)

(i) $\Longleftrightarrow$ (iii)

$$\phi_n \cdot S_n = \phi_{n+1} \cdot S_n \quad \forall n \in \{0, \dots, N-1\}$$

$$\Updownarrow \qquad \beta_n := \frac{1}{S_n^0}$$

$$\phi_n \cdot \underbrace{(\beta_n S_n)}_{\tilde{S}_n} = \phi_{n+1} \cdot \underbrace{(\beta_n S_n)}_{\tilde{S}_n}$$

$\Updownarrow$ - Remark (self-financing) for $\tilde{S}$ instead of S_n

$$\phi_{n+1} \cdot \tilde{S}_{n+1} = \phi_n \cdot \tilde{S}_n + \phi_{n+1} \cdot (\tilde{S}_{n+1} - \tilde{S}_n)$$

Hence,

$$\tilde{V}_{n+1}(\phi) \overset{(**)}{=} \tilde{V}_n(\phi) + \phi_{n+1} \cdot \Delta \tilde{S}_{n+1}$$

Now copy and paste the proof (i) $\Longleftrightarrow$ (ii) for $\tilde{S}_n$ instead of S_n, see also Ex. 4d) in Week 1

Δ

Remark:

Proposition 1 tells us that we can equivalently make our considerations in discounted or (on the first glance perhaps more naturally) in nominal (non-discounted) terms.

Note: We are working in a discrete time framework based on a finite probability space.

However, it will turn out that the advantage of considering discounted terms, and hence, the power of Proposition 1, is that discounted terms suite fine into martingale related considerations.

In view of the available machinery from martingale theory, this is a very considerable technical advantage.

Furthermore, note that

since

$$\Delta \tilde{S}_j^0 = \underbrace{\beta_j \cdot S_j^0}_{=1} - \underbrace{\beta_{j-1} S_{j-1}^0}_{=1} \qquad \left(\beta_n = \frac{1}{S_n^0} \ \forall n\right)$$

$$\Downarrow$$

The value of the discounted portfolio is completely defined by the initial wealth and the strategy

$$(\phi_n^1, \dots, \phi_n^d)_{0 \le n \le N},$$

cf. Proposition 2.

Proof of Proposition 2

$$\tilde{V}_n(\phi) = \phi_n^0 \cdot 1 + \dots + \phi_n^d \cdot \tilde{S}_n^d \underset{\text{Proposition 1 (III)}}{=} V_0 + \sum_{j=1}^{n} \phi_j \cdot \Delta\tilde{S}_j \quad \forall n \in \{1, \dots, N\}$$

$$\Updownarrow$$

$$\phi_n^0 = V_0 + \sum_{j=1}^{n-1} \phi_j \cdot \Delta\tilde{S}_j$$

$$+ \phi_n^1(\tilde{S}_n^1 - \tilde{S}_{n-1}^1) + \dots + \phi_n^d(\tilde{S}_n^d - \tilde{S}_{n-1}^d)$$

$$- \phi_n^1 \tilde{S}_n^1 \quad \dots \quad - \phi_n^d \tilde{S}_n^d$$

$$= V_0 + \sum_{j=1}^{n-1} \phi_j \cdot \Delta\tilde{S}_j$$

$$+ \phi_n^1(-\tilde{S}_{n-1}^1) + \dots + \phi_n^d(-\tilde{S}_{n-1}^d)$$

predictable

$\mathcal{F}_{n-1}$-measurable

---

i.e. ϕ^0 is predictable

and ϕ_n^0 is also uniquely

defined $\forall n \in \{1, \dots, N\}$.

Furthermore,

$$\tilde{V}_0(\phi) = V_0$$

$$\parallel$$

$$\phi_0^0 \cdot 1 + \phi_0^1 S_0^1 + \dots + \phi_0^d S_0^d$$

$$\Updownarrow$$

$$\phi_0^0 = \underbrace{V_0 - \phi_0^1 S_0^1 - \dots - \phi_0^d S_0^d}$$

$\mathcal{F}_0$-measurable and uniquely defined $\square$

Remark: • In the self-financing case

$$\phi_0 \cdot S_0 = \phi_1 \cdot S_0$$

$\uparrow$ investment for the period from 0 to 1

$\Downarrow$

ϕ_0 is not particularly interesting.

- We did not make any assumption on the sign of the quantities ϕ_n^i.

- If $\phi_n^0 < 0$ we have borrowed the amount $|\phi_n^0|$ in the riskless asset.

- If $\phi_n^i < 0$ for $i \geq 1$ we have a short position in asset i (where $|\phi_n^i|$ is the number)

- Here, short selling and borrowing are allowed.

- However: Value of the portfolio must remain non-negative at all times. $\rightsquigarrow$ Admissibility

$\Downarrow$

Able to pay back the debts at any time.

Remarks on martingales

- An $\mathbb{R}^d$-valued sequence of random vectors is a martingale if each component is a real-valued martingale.

0) $(M_n)_{0\le n\le N}$ - martingale iff

$$E(M_{n+j}|\mathcal{F}_n) = M_n \quad \forall j, \forall n$$

$$E(M_{n+j}|\mathcal{F}_n) = E(\underbrace{E(\ldots \underbrace{E(M_{j+n}|\mathcal{F}_{n+j-1})}_{=M_{n+j-1}}|\ldots}_{=M_{n+1}}|\mathcal{F}_n)$$

$$= M_n$$

(Conversely is a special case).

1) $$E(M_n) = E(\underbrace{\ldots \underbrace{E(M_n|\mathcal{F}_{n-1})}_{=M_{n-1}}|\ldots|\mathcal{F}_0)}_{=M_0}|\{\emptyset,\Omega\})$$

$$= E(M_0) \underset{\text{if } \mathcal{F}_0 = \{\emptyset,\Omega\}}{=} M_0$$

(more generally M_0 can be random)

3) $E(M^1_{j+1} + M^2_{j+1} | \mathcal{F}_j)$

$$\parallel$$

$$E(M^1_{j+1} | \mathcal{F}_j) + E(M^2_{j+1} | \mathcal{F}_j)$$

$$\parallel$$

$M^1_j + M^2_j$ for (M^1_n), (M^2_n) being martingales.

$$\Downarrow$$

The sum of two martingales is a martingale.

Proposition (Martingale Transform)

$\left.\begin{array}{l} \cdot\ (M_n)_{0\le n\le N} \text{ - martingale w.r.t. } (\mathcal{F}_n)_{0\le n\le N} \\ \cdot\ (H_n)_{0\le n\le N} \text{ - predictable sequence} \quad '' \quad '' \end{array}\right\} \Rightarrow$

$(X_n)_{0\le n\le N}$ defined by

$$X_n = \begin{cases} H_0 M_0 & \text{for } n=0 \\ H_0 M_0 + \sum_{i=1}^{n} H_i \underbrace{(M_i - M_{i-1})}_{\Delta M_i} & \text{for } n \ge 1 \end{cases}$$

is a martingale w.r.t. $(\mathcal{F}_n)_{0\le n\le N}$.

Remark: Of course, the martingale property also depends on the probability measure.

Proof (Martingale Transform)

- (X_n) - adapted is clear (composition of measurable functions)

- $$E(X_{n+1}-X_n \mid \mathcal{F}_n) = E(H_{n+1}(M_{n+1}-M_n) \mid \mathcal{F}_n)$$

$$= H_{n+1}\Big(\underbrace{E(M_{n+1}\mid\mathcal{F}_n)}_{= M_n} - \underbrace{E(\overset{\text{measurable}}{\overbrace{M_n}} \mid \mathcal{F}_n)}_{= M_n}\Big)$$

$\mathcal{F}_n$- measurable

martingale property of (M_n)

$$= 0$$

$$\Downarrow$$

$$E(X_{n+1}\mid\mathcal{F}_n) = \underbrace{E(\overset{\text{measurable}}{\overbrace{X_n}} \mid \mathcal{F}_n)}_{= X_n} = X_n$$

$\square$

Proof of Proposition 3

(characterisation of martingales;

(M_n)-mart. $\Longleftrightarrow E\left(\sum_{n=1}^{N} H_n \Delta M_n\right) = 0$

$\forall (H_n)$-pred.)

a) If (M_n) is a martingale

$\Downarrow$ Martingale Transform

(X_n) - defined by

$$X_n = \begin{cases} X_0 = 0 & \text{for } n = 0 \\ X_n = \sum_{j=1}^{n} H_j \Delta M_j & \text{for } n \geq 1 \end{cases}$$

for any (H_n)-predictable

is also a martingale.

$\Downarrow$

$$E(X_N) = E(X_0) = 0$$

$$\| $$

$$E\left(\sum_{j=1}^{N} H_j \Delta M_j\right)$$

$\Downarrow$

$$E\left(\sum_{j=1}^{N} H_j \Delta M_j\right) = 0$$

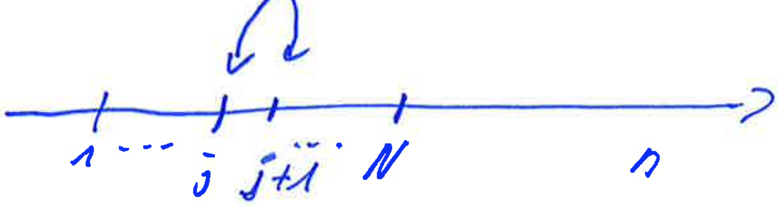
Conversely, if $E\left(\sum_{j=1}^{N} H_j \cdot \Delta M_j\right) \overset{(*)}{=} 0$

$\forall (H_n)$-predictable.

Then in particular for the sequences defined by

$$H_n = \begin{cases} 0 & \text{for } n \neq j+1 \\ \mathbb{1}_A & \text{for } n = j+1 \end{cases}$$

for any $\mathcal{F}_j$-measurable A



$\Downarrow$

(H_n)-predictable

Hence, (H_n)-predictable.

Thus,

$$E\left(\mathbb{1}_A (M_{j+1} - M_j)\right) \overset{(*)}{=} 0 \quad \forall A \in \mathcal{F}_j$$

Recall: For any integrable rv X (real valued) there exists a real valued integrable $\tilde{\mathcal{F}}$-measurable rv Y s.t.

$$\forall B \in \tilde{\mathcal{F}} \quad E(X \mathbb{1}_B) = E(Y \mathbb{1}_B);$$

sub-sigma algebra of $\mathcal{F}$

Y - a.s. unique. Notation $E(X|\tilde{\mathcal{F}}) = Y$

(conditional expectation)

Hence,

$$E(\mathbb{1}_A \underbrace{(M_{j+1} - M_j)}_{X}) = 0 = E(\mathbb{1}_A \cdot \underbrace{0}_{Y}) \quad \forall A \in \mathcal{F}_j$$

$$\Downarrow$$

$$E(M_{j+1} - M_j | \mathcal{F}_j) = 0$$

$$\Downarrow$$

$$E(M_{j+1} | \mathcal{F}_j) = E(M_j | \mathcal{F}_j) = M_j \quad \square$$

Proof of the FTAP I

(a) Assume that Q equivalent to P s.t. $\tilde{S}$ - Q (vector) martingale exists.

For ϕ admissible

$$\Downarrow \text{ Prop. 1 (iii)}$$

$$\tilde{V}_n(\phi) = V_0(\phi) + \sum_{j=1}^{n} \phi_j \cdot \Delta \tilde{S}_j \quad \text{- Martingale Transform}$$

$\Rightarrow (\tilde{V}_n(\phi))$ is a martingale

$$\Downarrow$$

$$E_Q(\tilde{V}_N(\phi)) = E_Q(\tilde{V}_0(\phi))$$

Hence, for an admissible ϕ with $V_0(\phi) = \tilde{V}_0(\phi) = 0$, we have

$$E_Q(\tilde{V}_N(\phi)) = 0 \quad \text{with} \quad \tilde{V}_N(\phi) \geq 0 .$$

Thus,

$$0 = E_Q(\tilde{V}_N(\phi)) = \sum_{\omega \in \Omega} \underset{\substack{\text{\rotatebox{90}{$\geq$}}\\ 0}}{\tilde{V}_N(\phi)(\omega)}\, \underset{\substack{\vee \\ 0}}{Q(\{\omega\})} \quad \forall \omega \in \Omega$$

$$\Downarrow$$

$$\tilde{V}_N(\phi) = 0, \quad Q\text{- a.s.}$$

$$\Downarrow$$

$Q(\tilde{V}_N(\phi) > 0) > 0$ is excluded

Since Q, P-equivalent $\Rightarrow P(\tilde{V}_N(\phi) > 0) > 0$ is excluded.

$$\Downarrow$$

Arbitrage is excluded.

(b) No-arbitrage $\Rightarrow \exists Q$ s.t. ...

Sketch:

1. Define $\Gamma = \{X :$ non-negative r.v., s.t. $P(X>0)>0\}$
   convex cone (in the space of r.v.)

   Hence, market viable $\Leftrightarrow \nexists\, \phi$-admissible with $V_0(\phi)=0 \Rightarrow \tilde{V}_N(\phi) \notin \Gamma$

2. ϕ-self-financing, define
   $$\tilde{G}_n(\phi) = \sum_{j=1}^{n} \phi_j \cdot \Delta \tilde{S}_j$$ - cumulative discounted gain process (for $V_0(\phi)=0$ $V_n(\phi)=G_n(\phi)$)

3. Lemma: If a market is viable, any <u>predictable</u> process $(\phi^1, \dots, \phi^d)$ satisfies $\tilde{G}_N(\phi) \notin \Gamma$

4. $V = \{$random variables $\tilde{G}_N(\phi)$: ϕ-predictable$\}$
   subspace of the space of r.v.

   $V \cap \Gamma = \emptyset$ - by Lemma and assumption (viable market)

   $\Downarrow$

   $\emptyset = V \cap K$, $\underset{\Gamma}{K \subset}$, $K = \{X \in \Gamma : \sum_\omega X(\omega) = 1\}$ - convex, compact.

⑤ $\Gamma \cap K = \emptyset \not\Rightarrow$ there is λ (rv)

Hahn-Banach (i) $\sum_\omega \lambda(\omega) X(\omega) > 0 \ \forall X \in K$

(ii) $\sum_\omega \lambda(\omega) \tilde{G}_\psi(\phi)(\omega) = 0 \ \forall \tilde{G}_\psi(\phi) \in V$

$\Downarrow$

construction of a probability measure Q

⑥ Q is equivalent to P

⑦ Q is a martingale measure

Details:

① Recall: A subset C of a vector space V is a convex cone if $\alpha X + \beta Y$ belongs to C for any $\alpha > 0$, $\beta > 0$ and any $X, Y \in C$.

For $\alpha > 0$, $X \in \Gamma$

$P(\alpha X > 0) > 0 \iff P(X > \frac{0}{\alpha}) > 0$

$\Downarrow$ for, $\alpha, \beta > 0, X, Y \in \Gamma$

$P(\alpha X + \beta Y > 0) \geq P(\alpha X > 0) > 0$

$\Downarrow$

$\alpha X + \beta Y \in \Gamma \ \forall \alpha, \beta > 0, \ X, Y \in \Gamma$

market viable $\iff \forall \phi$-admissible with

$V_0(\phi) = 0 \Rightarrow \tilde{V}_N(\phi) \notin \Gamma$

is a simple translation

② $((\phi_n^1, \dots, \phi_n^d))$-predictable, then

$\tilde{G}_n(\phi) = \sum_{j=1}^{n} \phi_j \cdot \Delta \tilde{S}_j$ - cumulative discounted gain process realised by following the self-financing strategy given by Proposition 2, starting at $V_0 = 0$

(Recall from Prop. 2: $\exists ! (\phi_n^0)$ s.t. $((\phi_n^0, \phi_n^1, \dots, \phi_n^d))$ is self-financing with $V_0(\phi) = \tilde{V}_0(\phi) = 0$)

For this strategy:

$\tilde{V}_n(\phi) = \tilde{G}_n(\phi)$

$\Downarrow$ ϕ-admissible

- $\tilde{G}_n(\phi) \geq 0 \quad \forall n = 1, \dots, N$ and
- $\tilde{G}_N(\phi) = 0$ - by assumption (viable market

3 Lemma: If a market is viable, any <u>predictable</u> process $(\phi^1, \dots, \phi^d)$ satisfies $\tilde{G}_N(\phi) \notin \Gamma$.

("get rid of the admissibility")

Proof of the Lemma

Assume by way of contradiction that

$$\tilde{G}_N(\phi) \in \Gamma$$

<u>First</u>: If $\tilde{G}_n(\phi) \geq 0 \ \forall n \in \{0, 1, \dots, N\}$ the market is obviously not viable.

<u>Second</u>: If the $\tilde{G}_n(\phi)$'s are not all non-negative, consider the last n such that

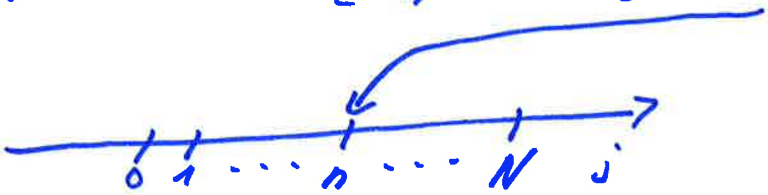
$$P(\tilde{G}_n(\phi) < 0) > 0.$$

Hence,
- $n \leq N-1$
- $P(\tilde{G}_n(\phi) < 0) > 0$
- $\forall m > n$, $\tilde{G}_m(\phi) \geq 0$ holds for this n.

Define the new process ψ by

$$\psi_j(\omega) = \begin{cases} 0 & \text{if } j \le n \\ \mathbb{1}_A(\omega)\,\phi_j(\omega) & \text{if } j > n \end{cases}$$

for $A = \{\tilde{G}_n(\phi) < 0\} \in \mathcal{F}_n$



$\Downarrow$

ψ-predictable with

$$\tilde{G}_j(\psi) = \begin{cases} 0 & \text{if } j \le n \\ \mathbb{1}_A\big(\underbrace{\tilde{G}_j(\phi)}_{\geq 0} \underbrace{- \tilde{G}_n(\phi)}_{> 0 \text{ on } A}\big) & \text{if } j > n \end{cases}$$

$\Downarrow$

$\tilde{G}_j(\psi) \geq 0$ for all $j \in \{0, \dots, N\}$

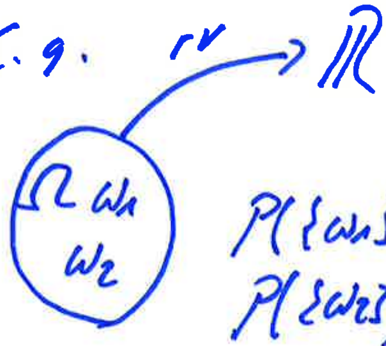
$\tilde{G}_N(\psi) > 0$ on A (!)

(4) $\mathbb{R}^{\Omega} = \{\Omega \longrightarrow \mathbb{R},\ rv\}$

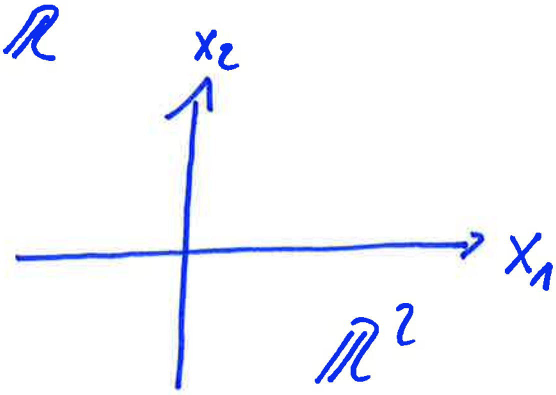
space of random variables defined on our probability space:

$$\begin{array}{ccc} \omega_1 & \longrightarrow & x_1 \\ \omega_2 & \longrightarrow & x_2 \\ & \vdots & \\ \omega_n & \longrightarrow & x_n \\ \cap & & \cap \\ \Omega & & \mathbb{R} \end{array}$$

E.g.



$P(\{\omega_1\})$
$P(\{\omega_2\})$



Hence, here the non-negative r.v. correspond to $\mathbb{R}^2_+$ and

$\Gamma = \{X : \text{non-neg, s.t. } P(X>0)>0\}$

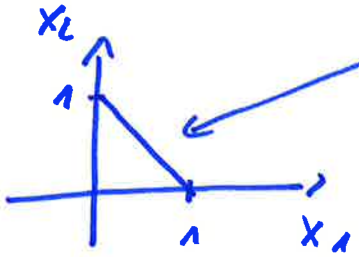
corresponds to $\mathbb{R}^2_+ \setminus \{0\}$.

Then

$K = \{X \in \mathbb{R}^{\Omega} : X(\omega) \geq 0,\ \forall \omega,\ \sum_{\omega} X(\omega) = 1\}$

is the unite simplex in the Eucledian space - convex and compact.

In the example $x_1 + x_2 = 1 \iff x_2 = 1 - x_1$



unit simplex

· Convex: $c \in [0,1]$, $Z = c\overset{\in K}{X} + (1-c)\overset{\in K}{Y}$:

$$\sum_\omega Z(\omega) = \sum_\omega cX(\omega) + \sum_\omega (1-c)Y(\omega)$$

$$= c\underbrace{\sum_\omega X(\omega)}_{=1} + \underbrace{\sum_\omega Y(\omega)}_{=1} - c\underbrace{\sum_\omega Y(\omega)}_{=1} = 1$$

i.e. in the simplex.

· V - subspace of $\mathbb{R}^n$ - clear

· $\tilde{G}_N(\phi) + \tilde{G}_N(\hat{\phi}) = \sum_{j=1}^N \phi_j \cdot \Delta\tilde{S}_j + \sum_{j=1}^N \hat{\phi}_j \cdot \Delta\tilde{S}_j$

$$= \sum_{j=1}^N (\phi_j + \hat{\phi}_j) \cdot \Delta\tilde{S}_j = \tilde{G}_N(\phi + \hat{\phi})$$

· $c \in \mathbb{R}$

$$c\tilde{G}_N(\phi) = c\sum_{j=1}^N \phi_j \cdot \Delta\tilde{S}_j = \sum_{j=1}^N (c\phi_j) \cdot \Delta\tilde{S}_j$$

$$= \tilde{G}_N(c\phi)$$

5 Lin.-algebra version of Hahn-Banach (convex separation theorem)

$$\left.\begin{array}{l}\mathbb{R}^n \supset K \text{ - convex, compact} \\ V \text{ - subspace of } \mathbb{R}^n\end{array}\right\} \Rightarrow$$

If $V \cap K = \phi$, then there is a linear function ξ defined on $\mathbb{R}^n$ satisfying

1) $\forall x \in K$, $\xi(x) > 0$
2) $\forall x \in V$, $\xi(x) = 0$

Therefore, the subspace V is included in a hyperplane that does not intersect K.

Hence, $V \cap K = \phi$

$\Downarrow$

there is a random variable λ, s.t.

(i) $\sum_\omega \lambda(\omega) X(\omega) > 0 \quad \forall X \in K$

(ii) $\sum_\omega \lambda(\omega) \tilde{G}_W(\phi)(\omega) = 0 \quad \forall \tilde{G}_W(\phi) \in V$

Note that $\mathbb{1}_{\omega_i}$ is also in particular in the unit simplex K.

Hence, $\sum_{\omega} \lambda(\omega) X(\omega) > 0 \ \forall X \in K$

$$\Downarrow - (X = \mathbb{1}_{\omega_i} \ \forall \omega_i)$$

$$\lambda(\omega) > 0 \ \forall \omega \in \Omega$$

Thus,

$$Q(\{\omega\}) := \frac{\lambda(\omega)}{\sum_{\omega' \in \Omega} \lambda(\omega')} \overset{6}{>} 0$$

$\forall \omega \in \Omega$, where

$$\sum_{\omega} Q(\{\omega\}) = \frac{\sum \lambda(\omega)}{\sum \lambda(\omega')} = 1$$

defines a probability measure equivalent to P.

7 Since by (ii) in Hahn-Banach

$$\sum_{\omega} \lambda(\omega) \tilde{G}_N(\phi)(\omega) = 0$$ for any predictable ϕ

$$\Downarrow$$

$$\sum_{\omega} \frac{\lambda(\omega)}{\sum_{\omega' \in \Omega} \lambda(\omega')} \tilde{G}_N(\phi)(\omega) = 0$$

$$\|$$

$$E_Q(\tilde{G}_N(\phi)) = \underbrace{E_Q\left(\sum_{j=1}^{N} \phi_j \cdot \Delta \tilde{S}_j\right)}_{} = 0.$$

$$\|$$

$$E_Q\left(\sum_{j=1}^{N} (\phi_j^1 \Delta \tilde{S}_j^1 + \dots + \phi_j^d \Delta \tilde{S}_j^d)\right)$$

for any predictable ϕ.

Hence, in particular for

$$(\underbrace{(0, \dots, 0, \phi_j^k, 0, \dots, 0)}_{\text{predictable}})$$

$$\Downarrow$$

$$E_Q\left(\sum_{j=1}^{N} \phi_j^k \Delta \tilde{S}_j^k\right) = 0 \quad \forall k \in \{1, \dots, d\}$$

$\Downarrow$ - Proposition 3

$(\tilde{S}_n^1), \dots, (\tilde{S}_n^d)$ - are Q-martingales $\square$

Remark:

Assume that a market is viable. Hence,

$$\left(\left(\tilde{S}_n^1, \ldots, \tilde{S}_n^d\right)\right) = \left(\left(\frac{S_n^1}{S_n^0}, \ldots, \frac{S_n^d}{S_n^0}\right)\right)$$

is a Q-martingale for at least one existing equivalent martingale measure Q.

Choose Q. Then

claim / payoff derivative

$$V_n \overset{(*)}{=} S_n^0 E_Q\left(\frac{X}{S_N^0} \Big| \mathcal{F}_n\right)$$

$$\Downarrow$$

$$\tilde{V}_n = \frac{V_n}{S_n^0} = E_Q\left(\frac{X}{S_N^0} \Big| \mathcal{F}_n\right) = E_Q(\tilde{X} | \mathcal{F}_n)$$

$\Downarrow$ - tower property ①

$(\tilde{V}_n)$ - Q-martingal

$$\Downarrow$$

$$\left(\left(\tilde{S}_n^1, \ldots, \tilde{S}_n^d, \tilde{V}_n\right)\right) \text{ - } Q\text{-martingal}$$

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① $E_Q(\tilde{V}_{n+1} | \mathcal{F}_n) = E_Q(\underbrace{E_Q(\tilde{X} | \mathcal{F}_{n+1})}_{\tilde{V}_{n+1}} | \mathcal{F}_n) = E_Q(\tilde{X} | \mathcal{F}_n) = \tilde{V}_n$

Thus, (*)

$$V_n = S_n^0 E_Q\left(\frac{X}{S_N^0} \,\middle|\, \mathcal{F}_n\right)$$ - is an arbitrage free pricing rule.

⚠ Q is not necessarily unique.