Extra notes: Comments on B&S

Slide 12 Some abuse of not.: F

$$C_0 = S_0 N\Bigg(\underbrace{\frac{\log\Big(\frac{\overbrace{S_0 e^{rT}}^{F}}{k}\Big) + \frac{1}{2}\sigma^2 T}{\sigma\sqrt{T}}}\Bigg) - k e^{-rT} N\Bigg(\underbrace{\frac{\log\Big(\frac{\overbrace{S_0 e^{rT}}^{F}}{k}\Big) - \frac{1}{2}\sigma^2 T}{\sigma\sqrt{T}}}\Bigg)$$

$$\frac{\log\left(\frac{F}{k}\right)}{\sigma\sqrt{T}} + \frac{1}{2}\sigma\sqrt{T} \qquad\qquad \frac{\log\left(\frac{F}{k}\right)}{\sigma\sqrt{T}} - \frac{1}{2}\sigma\sqrt{T}$$

$\sigma \downarrow 0$, $F>k$: $\underbrace{\frac{S_0}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\frac{1}{2}y^2}\,dy} = S_0$

$\sigma \downarrow 0$, $F=k$: $\underbrace{\frac{S_0}{\sqrt{2\pi}}\int_{-\infty}^{0} e^{-\frac{1}{2}y^2}\,dy} = S_0/2$

$\sigma \downarrow 0$, $F<k$: $\underbrace{\frac{S_0}{\sqrt{2\pi}}\int_{-\infty}^{-\infty} e^{-\frac{1}{2}y^2}\,dy} = 0$

$\sigma \downarrow 0$, $F>k$: $\underbrace{\frac{ke^{-rT}}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\frac{1}{2}y^2}\,dy} = ke^{-rT}$

$\sigma \downarrow 0$, $F=k$: $\underbrace{\frac{ke^{-rT}}{\sqrt{2\pi}}\int_{-\infty}^{0} e^{-\frac{1}{2}y^2}\,dy} = \frac{ke^{-rT}}{2}$

$\sigma \downarrow 0$, $F<k$: $\underbrace{\frac{ke^{-rT}}{\sqrt{2\pi}}\int_{-\infty}^{-\infty} e^{-\frac{1}{2}y^2}\,dy} = 0$

Hence, for $\sigma \downarrow 0$ we have

$F>k$: $C_0 = S_0 - ke^{-rT}$ reasonable ("in the money" without vola)

$F=k$: $C_0 = \frac{1}{2}(S_0 - ke^{-rT}) = \frac{1}{2}e^{-rT}(F-k) = 0$ reasonable ("at the money" without vola)

$F<k$: $C_0 = 0$ reasonable ("out of the money" without volatility).

- Note that we have obtained the lower static arbitrage bound.
- "In the money" etc. is here used including discounting effects (slightly non-standard)

$$C_0 = \underbrace{S_0 N\left(\frac{\log\left(\frac{F}{K}\right)}{\sigma\sqrt{T}} + \frac{1}{2}\sigma\sqrt{T}\right)}_{\substack{\downarrow\,\sigma\uparrow\infty \\ \frac{S_0}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\frac{y^2}{2}}\,dy \\ \parallel \\ S_0}} - \underbrace{Ke^{-rT} N\left(\frac{\log\left(\frac{F}{K}\right)}{\sigma\sqrt{T}} - \frac{1}{2}\sigma\sqrt{T}\right)}_{\substack{\downarrow\,\sigma\uparrow\infty \\ \frac{Ke^{-rT}}{\sqrt{2\pi}}\int_{-\infty}^{-\infty} e^{-\frac{y^2}{2}}\,dy \\ \parallel \\ 0}}$$

(independent of the sign of $\log\left(\frac{F}{K}\right)$)

Hence, for $\uparrow\infty$

$F > K$: $C_0 = S_0$ - upper static arbitrage bound

$F = K$: $C_0 = S_0$ " " " "

$F < K$: $C_0 = S_0$ " " " "

Recall $\max(S_0 - Ke^{-rT}, 0) \leq C \leq S_0$.

Hence, we have obtained the lower static arbitrage bound for $\sigma\downarrow 0$ and the upper static arbitrage bound for $\sigma\uparrow\infty$.

$$\Downarrow \text{ Vega} > 0$$

As long as the market price of the European Call is between the static arbitrage bounds, we can find an unique implicite volatility.

Inverse problems and model calibration

Consider a risk-neutral model with parameters $\theta \in \Theta$ - set of parameters (not necessarily only σ). Now:

Given observed prices C_i of call options for maturities T_i and strikes k_i.
Find θ such that the discounted asset price $(\tilde{S}_t)$ is a martingale and observed option prices are given by their risk-neutral expectations

$$\forall i \in I, \; C_i = e^{-rT_i} E_{Q^\theta}((S_{T_i} - k_i)_+)$$

This problem is usually ill-posed

- no solution at all may exist
- infinite number of solutions
- even with additional criterion to choose one solution, the dependence of the solution on input prices may be discontinuous $\Rightarrow$ numerical instability of the calibration algorithm.

Regularisation can sometimes help.

Roughly: Find $\theta \in \Theta$, which minimizes something like

$$\sum_{i=1}^{N} w_i \left| e^{-rT_i} E_{Q^\theta}(S_{T_i} - k_i)_+ - C_i \right|^2 + \underbrace{F(Q^\theta, P_0, \alpha)}$$

weight

additional penalty function depending on θ and P_0 and a parameter(s) α

subject to Q^θ to be equivalent to P_0 (and a martingale measure)

P_0 - estimated real-world measure

α - can be a parameter for fixing how much to regularise.