Extra notes Black & Scholes

Slide 5

$dS_t^0 = rS_t^0 dt$, $S_0^0 = 1$ means

$S_t^0 = S_0^0 + \int_0^t rS_s^0 ds$; or for given S_0^0

$$\frac{dS_t^0}{dt} = rS_t^0$$

$$S_t^0 = e^{rt}, \quad t \in [0,T]$$

Recall (see e.g. J.M. Steele, Stochastic Calculus and Financial Applications, Springer 2001, Sec. 8.4 - 8.7):

If a process (X_t) is given to us as a stochastic integral that we may write in shorthand as

$$dX_t = a(\omega,t)dt + b(\omega,t)dB_t$$

(B_t)-sBM

then it is natural to <u>define</u> the "dX_t-integral" of $f(\omega,t)$ by setting

$$\int_0^t f(\omega,s)\,dX_s = \int_0^t f(\omega,s)a(\omega,s)\,ds + \int_0^t f(\omega,s)b(\omega,s)\,dB_s$$

provided that $f(\omega,t)$ is an integrand for which the last two integrals make sense; i.e. $f(\omega,t)$ is adapted and

- $f(\omega,s)a(\omega,s) \in L^1(ds)$ $\forall \omega \in \bar{\Omega}$ with $P(\bar{\Omega}) = 1$
- $f(\omega,s)b(\omega,s) \in L^2_{loc} = L^2_{loc}[0,T]$

i.e. adapted, measurable, such that

$$P\left(\int_0^T (f(\omega,s)b(\omega,s))^2\,ds < \infty\right) = 1$$

Def.: We say that a process $(X_t)_{t\in[0,T]}$ is a standard process (Itô-process) provided that $(X_t)_{t\in[0,T]}$ has the integral representation

$$X_t = x_0 + \int_0^t a(\omega,s)\,ds + \int_0^t b(\omega,s)\,dB_s, \quad t \in [0,T]$$

where $a(\cdot,\cdot)$ and $b(\cdot,\cdot)$ are adapted measurable processes that satisfy the integrability conditions

$$P\left(\int_0^T |a(\omega,s)|\,ds < \infty\right) = 1 \quad \text{and}$$

$$P\left(\int_0^T |b(\omega,s)|^2\,ds < \infty\right) = 1.$$

Theorem $f \in C^{1,2}(\mathbb{R}_+ \times \mathbb{R})$ and $(X_t)_{t\in[0,T]}$ - standard process with the integral representation

$$X_t = x_0 + \int_0^t a(\omega,s)\,ds + \int_0^t b(\omega,s)\,dB_s, \quad 0 \le t \le T,$$

then

$$f(t,X_t) = f(0,x_0) + \int_0^t \frac{\partial f}{\partial s}(s,X_s)\,ds + \frac{1}{2}\int_0^t \frac{\partial^2 f}{\partial x^2}(s,X_s)\,b^2(\omega,s)\,ds + \underbrace{\int_0^t \frac{\partial f}{\partial x}(s,X_s)\,dX_s}_{=\int_0^t \frac{\partial f}{\partial x}(s,X_s)\,a(\omega,s)\,ds + \int_0^t \frac{\partial f}{\partial x}(s,X_s)\,b(\omega,s)\,dB_s}$$

Abbreviation

$$dX_t = a(\omega,t)\,dt + b(\omega,t)\,dB_t, \quad f \in C^{1,2}(\mathbb{R}_+ \times \mathbb{R})$$

$$\Downarrow$$

$$df(t,X_t) = \left(a(\omega,t)\frac{\partial f}{\partial x}(t,X_t) + \frac{\partial f}{\partial t}(t,X_t) + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(t,X_t)\,b^2(\omega,t)\right)dt$$

$$+\, b(\omega,t)\frac{\partial f}{\partial x}(t,X_t)\,dB_t$$

The risky asset in B&S is given by the SDE

$$dS_t = \mu S_t\,dt + \sigma S_t\,dB_t, \quad S_0 > 0.$$

sBM

constants.

σ is called the <u>volatility</u> of the asset

(annualised standard deviation of the log of the price-change factors)

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Exercises: Solution of the SDE
(unique for given S_0)

$$S_t = S_0 \exp(\mu t - \tfrac{1}{2}\sigma^2 t + \sigma B_t)$$

Spot price observed at time $t=0$

Recall (characteristic function)

$X \sim N(\mu, \sigma^2)$

$\varphi_X(u) = \exp(iu\mu - \tfrac{1}{2}\sigma^2 u^2)$, $u \in \mathbb{R}$,
can be extended to $\mathbb{C}$. (for $X \sim N(\mu,\sigma^2)$)
From that:

$$\underline{E(S_t)} = E(S_0 \exp(\mu t - \tfrac{1}{2}\sigma^2 t + \sigma B_t))$$

$$= S_0 \exp(\mu t - \tfrac{1}{2}\sigma^2 t) E(e^{i(-i)\sigma B_t})$$

$$= S_0 \exp(\mu t - \tfrac{1}{2}\sigma^2 t)\exp(-\tfrac{1}{2}(-i)^2\sigma^2 t)$$

$$= \underline{S_0 \exp(\mu t)}$$

expected yield

Filtrations Slide 6.

As far as filtrations are concerned it is often advisible to work with filtrations, sattisfying the "usual conditions".

Convenient since e.g.

- if $\mathcal{F}_t$ contains all P-null sets of $\mathcal{F}$ and if $X = Y$ P a.s. and Y is $\mathcal{F}_t$-measurable $\Rightarrow$ X is also $\mathcal{F}_t$-measurable
- If the filtration $(\mathcal{F}_t)_{t \in [0,T]}$ is right continuous it is possible to show that every martingale has a càdlàg version
- Advantages when working with stopping times, etc.

Recall: A process η is a version of ξ (sometimes: "modification") if

$$P(\xi(t) = \eta(t)) = 1 \quad \forall t \in T \text{ - considered time interval}$$

a version is indistinguishable if

$$P(\xi(t) = \eta(t), \forall t \in T) = 1$$

⚠ Continuity of sample paths does not imply continuity of filtrations.

Slide 8: Properties

$Y_t = \log(S_t)$, $dS_t = \underbrace{\mu S_t}_{a(\omega,t)} dt + \underbrace{\sigma S_t}_{b(\omega,t)} dB_t$

With a slight extension of Itô's formula to open subsets of $\mathbb{R}$

$$\frac{\partial f}{\partial x}(x) = \frac{1}{x}, \quad \frac{\partial^2 f}{\partial x^2}(x) = -\frac{1}{x^2}$$

$$\Downarrow$$

$$dY_t = \left(\frac{1}{S_t}\mu S_t + 0 + \frac{1}{2}\left(-\frac{1}{S_t^2}\right)\sigma^2 S_t^2\right)dt$$

$$\left[\text{Recall} \quad \left(\frac{\partial f}{\partial x}(S_t)a(\omega,t) + \frac{\partial f}{\partial t}(S_t) + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}b^2(\omega,t)\right)dt\right]$$

$$+ \frac{1}{S_t}\sigma S_t dB_t \quad = \left(\mu - \frac{1}{2}\sigma^2\right)dt + \sigma dB_t$$

$$\left[\text{Recall} \quad \frac{\partial f}{\partial x}(S_t)b(\omega,t)dB_t\right]$$

i.e.

$$Y_t = Y_0 + \int_0^t (\mu - \tfrac{1}{2}\sigma^2)\,ds + \int_0^t \sigma\, dB_s$$

$$= \log(S_0) + (\mu - \tfrac{\sigma^2}{2})t + \sigma B_t$$

$$\Downarrow$$

$$\log(S_t) \sim N(\log(S_0) + (\mu - \tfrac{1}{2}\sigma^2)t, \sigma^2 t)$$

Slide 11

Proof of Proposition 1

Recall first: Integration by parts for (X_t), (Y_t) - Itô-processes w.r.t. the same SBM (W_t)

$$X_t = x_0 + \int_0^t a(\omega,s)\,ds + \int_0^t b(\omega,s)\,dW_s$$

$$Y_t = y_0 + \int_0^t a'(\omega,s)\,ds + \int_0^t b'(\omega,s)\,dW_s$$

Then

$$X_t Y_t = x_0 y_0 + \int_0^t X_s\,dY_s + \int_0^t Y_s\,dX_s + \underbrace{\langle X,Y\rangle_t}_{\int_0^t b(\omega,s)b'(\omega,s)\,ds}$$

Suppose that ϕ is self-financing

$X_t = e^{-rt}$, $V_t(\phi) = Y_t$, $\tilde{V}_t(\phi) = e^{-rt} V_t(\phi)$

$$d\tilde{V}_t(\phi) = e^{-rt} dV_t(\phi) + V_t(\phi)\underbrace{(-re^{-rt})dt}_{d(e^{-rt})} + 0$$

ϕ-self-financing

$$= \cancel{e^{-rt} H_t^0 \underbrace{d(e^{rt})}_{re^{rt}dt}} + e^{-rt} H_t dS_t$$

$$\underbrace{(\cancel{H_t^0 e^{rt}} + H_t S_t)}_{V_t(\phi)}(-re^{-rt})dt$$

$$= H_t(e^{-rt} dS_t + (-re^{-rt})S_t dt)$$

$$= H_t\, d(e^{-rt} S_t)$$

$$= H_t\, d\tilde{S}_t$$

$$d(e^{-rt} S_t) = e^{-rt} dS_t + S_t(-re^{-rt})dt$$

i.e.

$$\tilde{V}_t(\phi) = V_0(\phi) + \int_0^t H_s\, d\tilde{S}_s \quad \text{a.s.}$$

The converse is obtained in a similar way

$$e^{rt}\tilde{V}_t(\phi) = V_t(\phi)$$

$$dV_t(\phi) = e^{rt}\, d\tilde{V}_t(\phi) + \tilde{V}_t(\phi)\, d(e^{rt}) + 0$$

by the "converse" assumption

$$= e^{rt} H_t\, d\tilde{S}_t + \tilde{V}_t(\phi)\, r e^{rt}\, dt$$

$$= e^{rt} H_t \left(e^{-rt} dS_t + S_t(-re^{-rt})dt \right) + rV_t(\phi)\,dt$$

$$= H_t\, dS_t - \cancel{H_t r S_t\, dt} + r(H^0_t e^{rt} + \cancel{H_t S_t})dt$$

$$= H_t\, dS_t + H^0_t\, d(\underbrace{e^{rt}}_{S^0_t})$$

i.e.

$$V_t(\phi) = V_0(\phi) + \int_0^t H^0_u\, dS^0_u + \int_0^t H_u\, dS_u \quad \text{a.s.} \quad \forall t \in [0,T]$$

□

Remark: Actually, it is still possible to define a predictable process in continuous time, but, in the case of the Brownian filtration, this does not restrict the class of adapted processes significantly.

Slide 12:

Remark: If Q is absolutely continuous with respect to P, with density Z, then P and Q are equivalent iff $P(Z>0)=1$

Slide 13:

Remark: A sufficient condition for $(L_t)_{t\in[0,T]}$ to be a martingale is

$$E\left(\exp\left(\frac{1}{2}\int_0^T \theta_t^2\,dt\right)\right)<\infty.$$

This is known as Novikov's condition.

(There are others, see e.g.
M. Mijatović, M. Urusov; On the martingale property of certain local martingales, Probability Theory and Related Fields, February 2012)

Slide 14 (Martingale Representation Th.)

Remark: Note that this representation only applies to martingales relative to the augmented filtration of the BM.

Remark: From the Martingale Representation Theorem it follows that if U is an $\mathcal{F}_T$-measurable, square-integrable random variable, it can be written as

$$U = E(U) + \int_0^T H_s \, dB_s \quad \text{a.s.}$$

where (H_t) is an adapted process such that $E\left(\int_0^T H_s^2 \, ds\right) < \infty$:

Consider
$$M_t = E(U \mid \mathcal{F}_t) = E(U) + \int_0^t H_s \, dB_s \quad \text{a.s.}$$

$$M_0 = E(U \mid \mathcal{F}_0) = E(U)$$

so that at T

$$M_T = E(U \mid \mathcal{F}_T) = U = E(U) + \int_0^T H_s \, dB_s$$

Remark: It can also be shown (see e.g. Karatzas and Shreve (1988)) that if $(M_t)_{t\in[0,T]}$ is (a not necessarily square-integrable) martingale, there is a similar representation with a process satisfying only $\int_0^T H_s^2 \, ds < \infty$ a.s.

Slide 15

From the SDE of (S_t) we have

$$d\tilde{S}_t = d(e^{-rt}S_t) = e^{-rt}dS_t + S_t(-re^{-rt})dt + 0$$

$$= e^{-rt}\mu S_t dt + e^{-rt}\sigma S_t dB_t - re^{-rt}S_t dt$$

$$= \tilde{S}_t((\mu - r)dt + \sigma dB_t)$$

$$= \tilde{S}_t \sigma \underbrace{\left(\frac{\mu - r}{\sigma} dt + dB_t \right)}_{d\tilde{W}_t}$$

Slide 16

Show that $(\tilde{S}_t)$ is a Q-martingale

(i) - adapted ✓

(ii) integrability, see (iii)

(iii) Assume that $s \leq t$

$$\underline{E_Q(\tilde{S}_t / \mathcal{F}_s)} = E_Q(\tilde{S}_0 e^{\sigma W_t + \overbrace{\sigma W_s - \sigma W_s}^{=0} - \frac{\sigma^2}{2} t} | \mathcal{F}_s)$$

$$\tilde{S}_0 \,\|\, S_0$$

$$= \tilde{S}_0 e^{\sigma W_s - \frac{\sigma^2}{2} t} \underbrace{E_Q(e^{\sigma (W_t - W_s)} | \mathcal{F}_s)}$$

$$\| $$

$$E_Q(e^{\sigma W_{t-s}}) \quad W_{t-s} \sim N(0, t-s)$$

$$\|$$

$$E_Q(e^{i(-i\sigma) W_{t-s}})$$

$$\|$$

$$\exp(-\tfrac{1}{2}(-i\sigma)^2 (t-s))$$

$$\|$$

$$\exp(\tfrac{1}{2}\sigma^2 (t-s))$$

$$= \tilde{S}_0 e^{\sigma W_s - \frac{\sigma^2}{2} s} = \underline{\tilde{S}_s}$$

Slide 17

Proof of Theorem 4

First assume that an admissible replicating strategy $\phi = (H^0, H)$ exists.

$\Downarrow$

$V_t(\phi) = H_t^0 S_t^0 + H_t S_t$ and by assumption, $V_T = h$.

Let $\tilde{V}_t(\phi) = V_t e^{-rt}$, i.e. $\tilde{V}_t(\phi) = H_t^0 + H_t \tilde{S}_t$.

Then by Prop. 1, $d\tilde{S}_t = \sigma \tilde{S}_t dW_t$, and the self-financing property of the strategy, we have

$$\tilde{V}_t(\phi) = V_0 + \int_0^t H_u \, d\tilde{S}_u$$

$$= V_0 + \int_0^t H_u \sigma \tilde{S}_u \, dW_u .$$

In view of the definition of admissible strategies we have that $(\tilde{V}_t(\phi))$ is a square-integrable Q-martingale.

Thus,

$$\tilde{V}_t(\phi) = E_Q(\tilde{V}_T(\phi) | \mathcal{F}_t) = E_Q\left(\frac{h}{e^{rT}} \Big| \mathcal{F}_t\right)$$

and consequently

$$V_t(\phi) = E_Q(e^{-r(T-t)} h | \mathcal{F}_t) = e^{-r(T-t)} E_Q(h | \mathcal{F}_t).$$

Hence, if an admissible portfolio (H^0, H) replicates the option defined by h, then $V_t = E_Q(e^{-r(T-t)} h | \mathcal{F}_t)$ is its value at t.

Thus, it remains to show that h is indeed replicable, i.e. we need to find an admissible strategy, such that

$$V_t(\phi) = H_t^0 S_t^0 + H_t S_t = E_Q(e^{-r(T-t)} h | \mathcal{F}_t)$$

$$= e^{rt} \underbrace{E_Q(e^{-rT} h | \mathcal{F}_t)}_{= M_t}$$

Note that $\forall t \in [0,T]$

$$E_Q(M_t^2) = E_Q(\underbrace{E_Q(e^{-rT}h/\mathcal{F}_t)}_{\text{def. } M_t}^2) \overset{\text{Jensen}}{\leq} E_Q(E_Q(e^{-2rT}h^2/\mathcal{F}_t)) = e^{-2rT}E_Q(h^2) < \infty$$

so that $(M_t)_{t\in[0,T]}$ is a square-integrable Q-martingale and that the augmented filtration of (B_t) is also the augmented filtration of (W_t).

$\Downarrow$ - Mart. Rep. Th. (Th. 3)

$\exists$ an adapted process $(K_t)_{t\in[0,T]}$ such that $E_Q(\int_0^T K_s^2 ds) < \infty$ and $\forall t \in [0,T]$, $M_t = M_0 + \int_0^t K_s dW_s$ a.s.

The strategy $\phi = (H^0, H)$

with

$$H_t = \frac{\kappa_t}{(\sigma \tilde{S}_t)} \quad \text{and} \quad H_t^0 = M_t - H_t \tilde{S}_t$$

is then self-financing:

$$\tilde{V}_t(\phi) = M_0 + \int_0^t \kappa_s \, dW_s$$

$$= M_0 + \int_0^t H_s \underbrace{\sigma \tilde{S}_s \, dW_s}_{d\tilde{S}_s}$$

$$= M_0 + \int_0^t H_s \, d\tilde{S}_s \text{, i.e. by}$$

Prop. 1 self-financing where

$$\underbrace{e^{rt}\tilde{V}_t(\phi)}_{\overset{\shortparallel}{M_t}} = V_t(\phi) = e^{rt} M_t = E_Q(e^{-r(T-t)} h / \mathcal{F}_t)$$

$$\shortparallel$$

Note

$$e^{rt} e^{-rT} E_Q(h/\mathcal{F}_t)$$

$$\shortparallel$$

$$e^{rt} e^{-rT} E_Q(h/\mathcal{F}_t) \overbrace{- H_t S_t + H_t S_t}^{0}$$

$$\underbrace{e^{rt}}_{S_t^0} \Big(\underbrace{e^{-rT} \overset{\shortparallel}{E_Q}(h/\mathcal{F}_t) - H_t \tilde{S}_t}_{H_t^0}\Big) + H_t S_t$$

Furthermore, note that
$\tilde{V}_t(\phi)$ is non-negative
(consequence of h being non-negative and the corresponding property of conditional expectations) and that

$$\infty > E_Q\left(\int_0^T K_s^2 \, ds\right) = E_Q\left(\int_0^T H_s^2 (\sigma \tilde{S}_s)^2 \, ds\right),$$

and clearly $V_T(\phi) = h$.

$\Downarrow$

There exists an admissible replication strategy.

$\square$

Slide 18

If we consider an European call defined by

$$h = (S_T - k)_+$$

Hence, $V_t = e^{-r(T-t)} E_Q((S_T - k)_+ | \mathcal{F}_t)$

$$= e^{-r(T-t)} E_Q\Big(\big(\underbrace{S_0 e^{(r-\frac{1}{2}\sigma^2)t+\sigma W_t}}_{\substack{S_t \\ \mathcal{F}_t\text{-measurable}}} \underbrace{e^{(r-\frac{1}{2}\sigma^2)(T-t)+\sigma(W_T-W_t)}}_{\substack{\text{price change} \\ \text{factor} \\ \text{independent}}} - k\big)_+ \Big| \mathcal{F}_t\Big)$$

Thus, $V_t = F(t, S_t)$, where

$$F(t,x) = e^{-r(T-t)} E_Q\left(\left(x e^{(r-\frac{1}{2}\sigma^2)(T-t)+\sigma(W_T-W_t)} - k\right)_+\right).$$

If $\sigma(W_T - W_t) + (r - \frac{1}{2}\sigma^2)(T-t) = Z + r(T-t)$

we have $Z \sim N(-\frac{1}{2}\sigma^2(T-t), \sigma^2(T-t))$.

Denote $\tau = T - t$

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$$F(t,x) = \frac{e^{-r\tau}}{\sqrt{2\pi\sigma^2\tau}} \int_{\mathbb{R}} \underbrace{\left(x e^{z+r\tau} - k\right)_+}_{\geq 0 \iff x e^{z+r\tau} \geq k \iff z \geq \log\left(\frac{k}{x}\right) - r\tau} e^{-\frac{\left(z+\frac{1}{2}\sigma^2\tau\right)^2}{2\sigma^2\tau}}\, dz$$

$$= \frac{1}{\sqrt{2\pi\sigma^2\tau}} \int_{\log\left(\frac{k}{x}\right)-r\tau}^{\infty} x e^{z} e^{-\frac{\left(z+\frac{1}{2}\sigma^2\tau\right)^2}{2\sigma^2\tau}}\, dz$$

$$- \frac{k e^{-r\tau}}{\sqrt{2\pi\sigma^2\tau}} \int_{\log\left(\frac{k}{x}\right)-r\tau}^{\infty} e^{-\frac{\left(z+\frac{1}{2}\sigma^2\tau\right)^2}{2\sigma^2\tau}}\, dz$$

$$v = -\frac{z+\frac{1}{2}\sigma^2\tau}{\sigma\sqrt{\tau}} \iff z = -\sigma\sqrt{\tau}\, v - \frac{1}{2}\sigma^2\tau$$

$$\frac{\log\left(\frac{x}{k}\right)+r\tau-\frac{1}{2}\sigma^2\tau}{\sigma\sqrt{\tau}} \qquad \left|\frac{dz}{dv}\right| = \sigma\sqrt{\tau}$$

$$= \frac{\sigma\sqrt{\tau}}{\sqrt{2\pi\sigma^2\tau}} \int_{-\infty}^{\left(\frac{\log\left(\frac{x}{k}\right)+\left(r-\frac{1}{2}\sigma^2\right)\tau}{\sigma\sqrt{\tau}}\right) =: d_-} x e^{-\sigma\sqrt{\tau}\, v - \frac{1}{2}\sigma^2\tau - \frac{1}{2}v^2}\, dv$$

$$- \frac{k e^{-r\tau}\sigma\sqrt{\tau}}{\sqrt{2\pi\sigma^2\tau}} \int_{-\infty} e^{-\frac{1}{2}v^2}\, dv$$

Note $-\frac{1}{2}\left(v+\sigma\sqrt{\tau}\right)^2 = -\frac{1}{2}v^2 - v\sigma\sqrt{\tau} - \frac{1}{2}\sigma^2\tau$

With $w = v + \sigma\sqrt{\tau}$, we have

$$\left|\frac{dv}{dw}\right| = 1$$

$$F(t,x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{d_- + \sigma\sqrt{\tau}} x e^{-\frac{1}{2}w^2}\, dw - \frac{e^{-r\tau}k}{\sqrt{2\pi}} \int_{-\infty}^{d_-} e^{-\frac{1}{2}v^2}\, dv$$

$$d_+ = d_- + \sigma\sqrt{\tau} = \frac{\log\left(\frac{x}{k}\right) + \left(r - \frac{1}{2}\sigma^2\right)\tau + \sigma^2\tau}{\sigma\sqrt{\tau}}$$

$$= \frac{\log\left(\frac{x}{k}\right) + \left(r + \frac{1}{2}\sigma^2\right)\tau}{\sigma\sqrt{\tau}}$$

$$= x\, N\left(\frac{\log\left(\frac{x}{k}\right) + \left(r + \frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{T-t}}\right)$$

$$- k e^{-r(T-t)} N\left(\frac{\log\left(\frac{x}{k}\right) + \left(r - \frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{T-t}}\right)$$

$$N(y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{y} e^{-\frac{1}{2}x^2}\, dx$$

Hence, $V_t = F(t, S_t)$

$$= S_t N\left(\frac{\log\left(\frac{S_t}{K}\right) + \left(r + \frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{T-t}}\right) - K e^{-r(T-t)} N\left(\frac{\log\left(\frac{S_t}{K}\right) + \left(r - \frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{T-t}}\right)$$

With the martingale property of $(\tilde{S}_t)$ we can easily get the formula for European put options.

$$e^{-r(T-t)} E_Q(S_T - k \mid \mathcal{F}_t) = \underbrace{e^{rt} E_Q(\tilde{S}_T - ke^{-rT} \mid \mathcal{F}_t)}$$

$(\mathcal{F}_t)$-mart

$$= \underbrace{e^{rt}\tilde{S}_t}_{\substack{\| \\ S_t}} - ke^{-r(T-t)}$$

$$= e^{rt}\left(E_Q((\tilde{S}_T - ke^{-rT})_+ \mid \mathcal{F}_t) - E_Q((ke^{-rT} - \tilde{S}_T)_+ \mid \mathcal{F}_t)\right)$$

$$\Downarrow$$

$$\underbrace{e^{-r(T-t)} E_Q((k - S_T)_+ \mid \mathcal{F}_t)}_{\text{put price}} = \underbrace{e^{-r(T-t)} E_Q((S_T - k)_+ \mid \mathcal{F}_t)}_{\text{known call price}} + ke^{-r(T-t)} - S_t$$

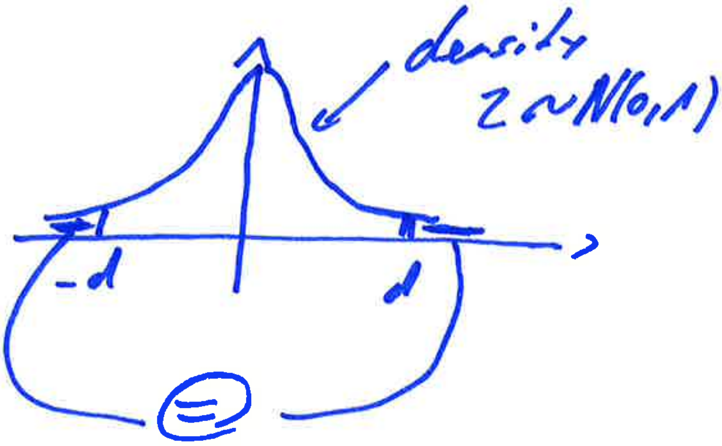
$$= \underbrace{S_t N\Big(\underbrace{\frac{\log(\frac{S_t}{k}) + (r + \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}}_{d_+}\Big) - ke^{-r(T-t)} N\Big(\underbrace{\frac{\log(\frac{S_t}{k}) + (r - \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}}_{d_-}\Big) - S_t + ke^{-r(T-t)}}$$

$$\|$$

$$ke^{-r(T-t)}(1 - N(d_-)) - S_t(1 - N(d_+))$$

Hence, we have for the European put

$$e^{-r(T-t)} E_Q((k-S_T)_+ \mid \mathcal{F}_t) = k e^{-r(T-t)} (1-N(d_-)) - S_t (1-N(d_+))$$



$$e^{-r(T-t)} E_Q((k-S_T)_+ \mid \mathcal{F}_t) = k e^{-r(T-t)} N(-d_-) - S_t N(-d_+)$$

This is the price of the European put with strike k and maturity T at time t.

Slide 19: Hedging

In the proof of Th. 4 we referred to the Martingale Representation Theorem to show the existence of a replicating portfolio/strategy. In practice, an existance theorem is not satisfactory for application and it is essential to be able to get an explicite real replication portfolio to hedge an option.

When the option is defined by a random variable $h = f(S_T)$, we can see that it is usually possible to find an explicite hedging portfolio, see below.

Consider an European claim (others will be considered later)

$$V_t = e^{-r(T-t)} E_Q\left(f(S_T) \mid \mathcal{F}_t\right)$$

$$= e^{-r(T-t)} E_Q\left(f\left(S_0 e^{(r-\frac{1}{2}\sigma^2)T + \sigma W_T}\right) \mid \mathcal{F}_t\right)$$

$$= e^{-r(T-t)} E_Q\left(f\left(\underbrace{S_0 e^{(r-\frac{1}{2}\sigma^2)t + \sigma W_t}}_{S_t} \underbrace{e^{(r-\frac{1}{2}\sigma^2)(T-t) + \sigma(W_T - W_t)}}_{\text{independent}}\right) \mid \mathcal{F}_t\right)$$

measurable

$\Downarrow$ Proposition from the exercises

$V_t = F(t, S_t)$, with

$$F(t,x) = e^{-r(T-t)} \int_{\mathbb{R}} f\left(x e^{(r-\frac{1}{2}\sigma^2)(T-t) + \sigma\sqrt{T-t}\, z}\right) \frac{e^{-\frac{z^2}{2}}}{\sqrt{2\pi}} \, dz$$

Note that by integrating the payoff function multiplied by the above density we obtain a "regularisationing effect".

Furthermore, recall that

$$S_t = S_0 \exp\left(\left(r - \tfrac{1}{2}\sigma^2\right)t + \sigma W_t\right)$$

for a Q-SBM (W_t) so that (Itô)

$$dS_t \overset{(*)}{=} \underbrace{rS_t}_{a_t}\,dt + \underbrace{\sigma S_t}_{b_t}\,dW_t$$

By Itô's formula

$$dF(t,S_t) = \left(rS_t \frac{\partial F}{\partial x}(t,S_t) + \frac{\partial F}{\partial t}(t,S_t) + \frac{1}{2}\sigma^2 S_t^2 \frac{\partial^2 F}{\partial x^2}(t,S_t)\right)dt$$

$$+ \left(\sigma S_t \frac{\partial F}{\partial x}(t,S_t)\right)dW_t$$

$$= dV_t(\phi)$$

ϕ-self-financing

$$= H_t^0\, d(\underbrace{e^{rt}}_{S_t^0}) + H_t\, dS_t$$

$$\overset{(*)}{=} H_t^0 r e^{rt} dt + H_t r S_t dt + H_t \sigma S_t dW_t$$

$$= (H_t^0 e^{rt} + H_t S_t) r\,dt + (\sigma S_t H_t)\,dW_t$$

$=$

Equality, since the starting point, (a_t) and (b_t) uniquely determine an Itô-process (standard process).

Respectively / more generally:
Decompositions of special semimartingales are unique, see Section 5; continuous semimartingales are special.

Thus,

$$H_t = \frac{\partial F}{\partial x}(t,S_t) \text{ - called } \Delta\text{-hedging}$$

Hence,

$$V_t(\phi) = F(t,S_t) = e^{-r(T-t)}E_Q(h|\mathcal{F}_t)$$

$$= \underbrace{e^{rt}}_{S_t^0}\left(e^{-rt}F(t,S_t) - H_t\tilde{S}_t\right) + H_tS_t$$

$$= S_t^0\Big(\underbrace{e^{-rt}\Big(F(t,S_t) - \frac{\partial F}{\partial x}(t,S_t)S_t\Big)}_{\substack{\shortparallel \\ H_t^0 \\ \shortparallel \\ \left(e^{-rT}E_Q(h|\mathcal{F}_t) - H_t\tilde{S}_t\right)}}\Big) + \underbrace{\frac{\partial F}{\partial x}(t,S_t)}_{H_t}S_t$$

$$\Downarrow$$

$$\phi_t = (H_t^0, H_t) = \left(e^{-rt}\Big(F(t,S_t) - \frac{\partial F}{\partial x}(t,S_t)S_t\Big), \frac{\partial F}{\partial x}(t,S_t)\right)$$

Alternatively,
(as on the slides) note that

$$M_t = \tilde{V}_t = e^{-rt} V_t = e^{-rt} F(t, S_t)$$

If we set $\tilde{F}(t,x) = e^{-rt} F(t, x e^{rt})$

$$\Downarrow$$

$$\tilde{V}_t = e^{-rt} F(t, S_t) = e^{-rt} F\left(t, \frac{S_t}{e^{rt}} e^{rt}\right)$$

$$= \tilde{F}(t, \tilde{S}_t)$$

and for $t < T$ we get from Itô's formula

$$\tilde{V}_t = \tilde{F}(t, \tilde{S}_t)$$

$$= \tilde{F}(0, \tilde{S}_0) + \int_0^t \overbrace{\left(\frac{\partial \tilde{F}}{\partial u}(u, \tilde{S}_u) + \frac{1}{2} \sigma^2 \tilde{S}_u^2 \frac{\partial^2 \tilde{F}}{\partial x^2}(u, \tilde{S}_u) \right)}^{=: K_u} du$$

$$+ \int_0^t \sigma \tilde{S}_u \frac{\partial \tilde{F}}{\partial x}(u, \tilde{S}_u)\, dW_u$$

where the martingale property of $\tilde{F}(t, \tilde{S}_t) = \tilde{V}_t = M_t$ implies $K_u = 0$

(decompositions of special semimartingales are unique, see Section 5; continuous semimartingales are special)

$$\tilde{F}(t,x) = e^{-rt} F(t, xe^{rt})$$

$$\Downarrow$$

$$\frac{\partial \tilde{F}}{\partial x}(t,x) = e^{-rt} \frac{\partial F}{\partial x}(t, xe^{rt}) e^{rt} = \frac{\partial F}{\partial x}(t, xe^{rt})$$

$$\Downarrow$$

$$\frac{\partial \tilde{F}}{\partial x}(t,\tilde{S}_t) = \frac{\partial F}{\partial x}(t, \tilde{S}_t e^{rt}) = \frac{\partial F}{\partial x}(t, S_t)$$

Hence,

$$V_t(\phi) = \underbrace{e^{rt}}_{S_t^0} \overbrace{\big(\underbrace{e^{-rT} E_Q(h \mid \mathcal{F}_t)}_{e^{-rt} F(t,S_t)} - H_t \tilde{S}_t\big)}^{H_t^0} + H_t S_t$$

$$= S_t^0 \left(e^{-rt}(F(t,S_t) - H_t S_t)\right) + H_t S_t$$

$$= S_t^0 \underbrace{\left(e^{-rt}\left(F(t,S_t) - \frac{\partial F}{\partial x}(t,S_t) S_t\right)\right)}_{H_t^0} + \underbrace{\frac{\partial F}{\partial x}(t,S_t)}_{H_t} S_t$$

Called Δ-hedging

$$= F(t,S_t)$$