Exercises; extra material W9

1) B&S-setting. Compute the forward price.

For $k \in \mathbb{R}_+$ we always have

$$(S_T - k) = (S_T - k)_+ - (k - S_T)_+$$ i.e. the difference of two claims;

k should be fixed at t such that there is no initial CF if a contract is entered at t. Hence,

$$e^{-r(T-t)} E_Q((S_T - k) \mid \mathcal{F}_t) = e^{rt} \underbrace{E_Q\Big(\underbrace{\frac{S_T}{e^{rT}}}_{\tilde{S}_T} \Big| \mathcal{F}_t\Big)}_{\tilde{S}_t} - k e^{-r(T-t)}$$

$$= S_t - \underbrace{k}_{F_{t,T}} e^{-r(T-t)} \overset{!}{=} 0$$ For the forward price at t

$$\Downarrow$$

$$F_{t,T} = S_t e^{r(T-t)}, \quad t \in [0, T]$$

$$F_{t,T} = S_0 e^{(r-\frac{1}{2}\sigma^2)t + \sigma W_t + r(T-t)}$$

$$= \underbrace{S_0 e^{rT}}_{F_{0,T}} \cdot e^{-\frac{1}{2}\sigma^2 t + \sigma W_t}$$

$$\parallel$$

$$X_t, \quad \text{s.t. } dX_t = \underbrace{-\tfrac{1}{2}\sigma^2}_{a_t} dt + \underbrace{\sigma}_{b_t} dW_t$$

$$F(X_t) = F_{0,T} e^{X_t} = F_{t,T}, \quad F(x) = F_{0,T} e^x$$

$$\frac{\partial F}{\partial x}(x) = F(x) = \frac{\partial^2 F}{\partial x^2}(x), \quad \frac{\partial F}{\partial t}(x) = 0$$

$$\Downarrow \text{Itô}$$

$$\underline{dF_{t,T}} = \left(-\tfrac{1}{2}\sigma^2 F_{t,T} + 0 + \tfrac{1}{2}\sigma^2 F_{t,T}\right)dt + \sigma F_{t,T} dW_t$$

$$= \underline{\sigma F_{t,T} dW_t}$$

2) PCS: For $t \in [0,T]$ write

$$S_T = \underbrace{\underbrace{S_0 e^{(r-\frac{1}{2}\sigma^2)t+\sigma W_t}}_{S_t} e^{r(T-t)}}_{F_{t,T}} \underbrace{e^{-\frac{1}{2}\sigma^2(T-t)+\sigma(W_T-W_t)}}_{Y_{t,T}}$$

$$\left(\text{Exponent: } \cancel{rt} - \cancel{\tfrac{1}{2}\sigma^2 t} + \cancel{\sigma W_t} + rT - \cancel{rt} - \tfrac{1}{2}\sigma^2 T + \cancel{\tfrac{1}{2}\sigma^2 t} + \sigma W_T - \cancel{\sigma W_t} = (r-\tfrac{1}{2}\sigma^2)T + \sigma W_T\right)$$

$$C_t = e^{-r(T-t)} E_Q((S_T-k)_+ \mid \mathcal{F}_t)$$
$$= S_t N(d_+) - k e^{-r(T-t)} N(d_-)$$

$$P_t = e^{-r(T-t)} E_Q((k-S_T)_+ \mid \mathcal{F}_t)$$
$$= k e^{-r(T-t)} N(-d_-) - S_t N(-d_+)$$

with

$$d_\pm = \frac{\log\left(\frac{S_t}{k}\right) + (r \pm \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}} = \frac{\log\left(\frac{S_t e^{r(T-t)}}{k}\right) \pm \frac{\sigma^2}{2}(T-t)}{\sigma\sqrt{T-t}}$$

$$= \frac{\log\left(\frac{F_{t,T}}{k}\right) \pm \frac{\sigma^2}{2}(T-t)}{\sigma\sqrt{T-t}}.$$

(11)

$$P_t = e^{-r(T-t)}\left(k\,N\left(-\frac{\log\left(\frac{F_{t,T}}{k}\right)-\frac{\sigma^2}{2}(T-t)}{\sigma\sqrt{T-t}}\right)-F_{t,T}\,N\left(-\frac{\log\left(\frac{F_{t,T}}{k}\right)+\frac{\sigma^2}{2}(T-t)}{\sigma\sqrt{T-t}}\right)\right)$$

On the other hand

$$C_t = e^{-r(T-t)}\left(F_{t,T}\,N\left(\frac{\log\left(\frac{F_{t,T}}{k}\right)+\frac{\sigma^2}{2}(T-t)}{\sigma\sqrt{T-t}}\right)-k\,N\left(\frac{\log\left(\frac{F_{t,T}}{k}\right)-\frac{\sigma^2}{2}(T-t)}{\sigma\sqrt{T-t}}\right)\right)$$

$$= e^{-r(T-t)}\left(F_{t,T}\,N\left(-\frac{\log\left(\frac{k}{F_{t,T}}\right)-\frac{\sigma^2}{2}(T-t)}{\sigma\sqrt{T-t}}\right)-k\,N\left(-\frac{\log\left(\frac{k}{F_{t,T}}\right)+\frac{\sigma^2}{2}(T-t)}{\sigma\sqrt{T-t}}\right)\right)$$

i.e. a European put price with strike $F_{t,T}$ and forward price k.

$$\Downarrow$$

$$e^{-r(T-t)}E_Q((S_T-k)_+|\mathcal{F}_t) = e^{-r(T-t)}E_Q((\underbrace{\overbrace{F_{t,T}\,\Lambda_{t,T}}^{\text{independent}}-k)_+|\mathcal{F}_t}_{\text{measurable}})$$

$$= e^{-r(T-t)}E_Q((F_{t,T}-k\Lambda_{t,T})_+|\mathcal{F}_t)$$

Of course, this equality can also be shown in detail with our proposition for conditional expectations for functions on random variables, where one is $\mathcal{F}_t$-measurable and the other one is independent of $\mathcal{F}_t$. This Proposition is also used for the above valuation functions we have applied and rewritten

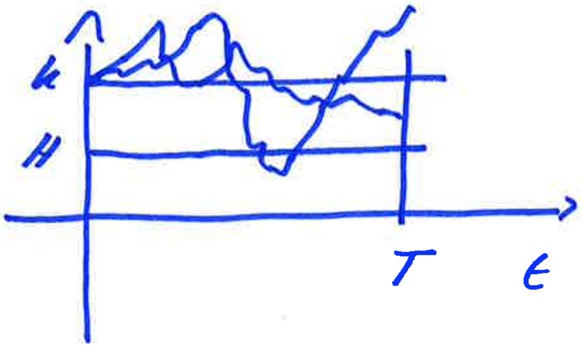
By positive homogeneity we further have

$$e^{-r(T-t)}E_Q((F_{t,T}\mathbb{1}_{t,T}-k)_+|\mathcal{F}_t) \overset{\text{before}}{=} e^{-r(T-t)}E_Q((F_{t,T}-k\mathbb{1}_{t,T})_+|\mathcal{F}_t)$$

$$= e^{-r(T-t)}\frac{k}{F_{t,T}}E_Q\left(\left(\frac{F_{t,T}^2}{k} - \underbrace{F_{t,T}\mathbb{1}_{t,T}}_{S_T}\right)_+\Big|\mathcal{F}_t\right)$$

$$= e^{-r(T-t)}\frac{k}{F_{t,T}}E_Q\left(\left(\frac{F_{t,T}^2}{k} - S_T\right)_+\Big|\mathcal{F}_t\right)$$

3)



$$X_{doc} = (S_T-k)_+\mathbb{1}_{\{S_t > H,\ \forall t\in[0,T]\}}$$

Hedge: Long: $(S_T-k)_+$

short $\frac{k}{H}\left(\frac{H^2}{k}-S_T\right)_+$

1-Case: Barrier avoided

$$\Downarrow$$

$$S_T > H \text{ and also } H < k \Leftrightarrow \frac{H}{k} < 1$$

$$\Downarrow$$

$$\frac{H}{k} H - S_T \leq H - S_T < 0$$

$$\Downarrow$$

puts expire worthless and $(S_T - k)_+$-replicates the X_{doc}.

2-Case: Barrier hit

We use without proof that the European PCS also holds for $[0,T]$-valued stopping times (consequence of the strong Markov property of BM).

Consider $\hat{\tau} = \inf\{t : S_t = H\}$ and $\tau = \hat{\tau} \wedge T$.

On $\{\hat{\tau} \leq T\}$ we have $S_{\tau} = H$ (since (S_t) is continuous) and for vanishing cc. $S_t = F_{t,T}$ $\forall t$

$$e^{-r(T-\tau)}\overbrace{E_Q\left(\left(S_\tau 1\!\!1_{\tau,T}-k\right)_+ \mid \mathcal{F}_\tau\right)}^{\text{measurable}}$$

$$\parallel$$

$$e^{-r(T-\tau)}E_Q\left(\left(H 1\!\!1_{\tau,T}-k\right)_+ \mid \mathcal{F}_\tau\right)$$

$$\parallel$$

$$e^{-r(T-\tau)}E_Q\left(\left(H-k 1\!\!1_{\tau,T}\right)_+ \mid \mathcal{F}_\tau\right)$$

$$\parallel$$

$$e^{-r(T-\tau)}\frac{k}{H}E_Q\left(\left(\frac{H^2}{k}-H 1\!\!1_{\tau,T}\right)_+ \mid \mathcal{F}_\tau\right)$$

$$\parallel$$

$$e^{-r(T-\tau)}\frac{k}{H}E_Q\left(\left(\frac{H^2}{k}-S_T\right)_+ \mid \mathcal{F}_\tau\right).$$

Hence, long and short position have the same value $\Rightarrow$ portfolio can be liquidated at zero costs.

4) Decomposition of European payoff functions

$f: \mathbb{R}_+ \longrightarrow \mathbb{R}$ two times continuously differentiable

Fundamental Th. of c.

$$\int_a^x f'(k)\,dk = f(x) - f(a). \quad \text{Hence,}$$

$$f(x) = f(a) + \int_a^x f'(k)\,dk = f(a) + xf'(x) - af'(a) - \int_a^x kf''(k)\,dk$$

Partial integration:

$$\int_a^x 1 \cdot f'(k)\,dk = kf'(k)\Big|_a^x - \int_a^x kf''(k)\,dk$$

$$= xf'(x) - af'(a) - \int_a^x kf''(k)\,dk$$

$$x\int_a^x f''(k)\,dk = xf'(x) - xf'(a)$$

$$\Longleftrightarrow$$

$$xf'(x) = xf'(a) + x\int_a^x f''(k)\,dk$$

$$= f(a) + x\int_a^x f''(k)\,dk + xf'(a) - af'(a) - \int_a^x kf''(k)\,dk$$

$$= f(a) + f'(a)(x-a) + \int_a^x f''(k)(x-k)\,dk$$

⑨

$$f(x) = f(a) + f'(a)(x-a) + \int_a^x f''(k)(x-k)_+ dk - \int_a^x f''(k)(k-x)_+ dk$$

$$= f(a) + f'(a)(x-a) + \int_a^x f''(k)(x-k)_+ dk + \int_x^a f''(k)(k-x)_+ dk$$

note that for $k > x \Rightarrow (x-k)_+ = 0$

for $k < x \Rightarrow (k-x)_+ = 0$

$$= f(a) + f'(a)(x-a) + \int_a^\infty f''(k)(x-k)_+ dk + \int_0^a f''(k)(k-x)_+ dk$$

5) Implicite distribution

$k > 0$

(undiscounted Europ. Put-call parity)

$$E_Q((S_T-k)_+) = S_0 e^{rT} - k + E_Q((k-S_T)_+)$$

$$= S_0 e^{rT} - k + \int_0^k (k-x) q(x) dx$$

$$= S_0 e^{rT} - k + k\int_0^k q(x)dx - \int_0^k x q(x) dx$$

$\Downarrow \partial/\partial k$ product rule

$$\frac{\partial}{\partial k}(E_Q((S_T-k)_+)) = -1 + \int_0^k q(x)dx + \cancel{k q(k)} - \cancel{k q(k)}$$

$$= -1 + \int_0^k q(x)dx$$

$\Downarrow \partial/\partial k$

$$\frac{\partial^2}{\partial k^2}(E_Q((S_T-k)_+)) = q(k), \qquad k > 0.$$

Extra remark on PCS

$\eta > 0$ a.s.

$$E_Q((F\eta - k)_+) = E_Q((F - k\eta)_+), \quad \forall (F,k) \in \mathbb{R}_+^2$$

Note $(1,0) \in \mathbb{R}_+^2$, hence

$$\underline{E_Q(\eta)} = E_Q((1\eta - 0)_+) = E_Q((1-0)_+) = \underline{1}.$$

Hence, we can define a new probab. measure by

$$\frac{d\tilde{Q}}{dQ} = \eta. \qquad \text{Denote } \tilde{\eta} = \frac{1}{\eta}$$

Since

$$E_{\tilde{Q}}((F\tilde{\eta} - k)_+) = E_{\tilde{Q}}((F\tfrac{1}{\eta} - k)_+)$$

$$= E_Q((F\tfrac{1}{\eta} - k)_+ \cdot \eta) = E_Q((F - k\eta)_+)$$

$$\overset{PCS}{=} E_Q((F\eta - k)_+) \quad \forall F,k \geq 0$$

$\Downarrow$ "Ex. 5" - (more general version)

η under Q $\overset{d}{=}$ $\tilde{\eta}$ under $\tilde{Q}$

$$E_Q(f(\eta)) = E_{\tilde{Q}}\left(f(\eta)\cdot\frac{1}{\eta}\right)$$
$$= E_{\tilde{Q}}\left(f\left(\frac{1}{1/\eta}\right)\cdot\frac{1}{\eta}\right)$$
$$= E_{\tilde{Q}}\left(f\left(\frac{1}{\tilde{\eta}}\right)\tilde{\eta}\right)$$
$$\stackrel{\ast}{=} E_Q\left(f\left(\frac{1}{\eta}\right)\eta\right)$$

η under Q
$\stackrel{d}{=}$
$\tilde{\eta}$ under $\tilde{Q}$

Hence, $\forall\, f: (0,\infty) \longrightarrow \mathbb{R}_+$ - measurable

$$E_Q(f(\eta)) = E_Q\left(f\left(\frac{1}{\eta}\right)\eta\right)$$

i.e. an at the first glance much stronger symmetrie is implied.