Exercises; extra material W8

1) $\tau = T - t$

$$C(\tau, S_t, K, r, \sigma) = S_t N(d_+) - Ke^{-r\tau} N(d_-)$$
$$P(\tau, S_t, K, r, \sigma) = Ke^{-r\tau}(1 - N(d_-)) - S_t(1 - N(d_+))$$

$$d_\pm \quad \frac{\log\left(\frac{S_t}{K}\right) + \left(r \pm \frac{1}{2}\sigma^2\right)\tau}{\sigma\sqrt{\tau}}$$

Tech. Lemma: $Ke^{-r\tau}\varphi(d_-) = \underbrace{S_t}_{S \text{ - abuse of notation}}\varphi(d_+)$

$$\varphi(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2}$$

Proof:

$$\varphi(d_-) = \frac{1}{\sqrt{2\pi}} \exp\Bigg(-\frac{1}{2}\Bigg(\overbrace{\frac{\log\left(\frac{S}{K}\right) + \left(r + \frac{1}{2}\sigma^2\right)\tau}{\sigma\sqrt{\tau}}}^{d_+} - \sigma\sqrt{\tau}\Bigg)^2\Bigg)$$

$$= \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}d_+^2 + \log\left(\frac{S}{K}\right) + r\tau + \cancel{\frac{1}{2}\sigma^2\tau} - \cancel{\frac{1}{2}\sigma^2\tau}\right)$$

$$= \varphi(d_+)\frac{S}{K}e^{r\tau}$$

$$\Updownarrow$$

$$\boxed{\varphi(d_-)Ke^{-r\tau} = \varphi(d_+)S}$$

Call

$$\underline{\Delta} = \frac{\partial V}{\partial S} = \frac{\partial}{\partial S}\left(S N\Big(\underbrace{\frac{\log\left(\frac{S}{K}\right) + \left(r + \frac{1}{2}\sigma^2\right)\tau}{\sigma\sqrt{\tau}}}_{d_+}\Big) - K e^{-r\tau} N\Big(\underbrace{\frac{\log\left(\frac{S}{K}\right) + \left(r - \frac{1}{2}\sigma^2\right)\tau}{\sigma\sqrt{\tau}}}_{d_-}\Big)\right)$$

$$= N(d_+) + S\varphi(d_+)\frac{\partial d_+}{\partial S} - K e^{-r\tau}\varphi(d_-)\frac{\partial d_-}{\partial S}$$

$$= N(d_+) + \cancel{S}\varphi(d_+)\frac{1}{\sigma\sqrt{\tau}}\frac{\cancel{K}}{\cancel{S}}\cdot\frac{1}{\cancel{K}} - \underbrace{K e^{-r\tau}\varphi(d_-)}_{\substack{\text{II - Lemma} \\ S\varphi(d_+)}}\frac{1}{\sigma\sqrt{\tau}}\frac{\cancel{K}}{\cancel{S}}\cdot\frac{1}{\cancel{K}}$$

$$= N(d_+) + \varphi(d_+)\frac{1}{\sigma\sqrt{\tau}} - \varphi(d_+)\cdot\frac{1}{\sigma\sqrt{\tau}}$$

$$= \underline{N(d_+)}$$

Put

$$\underline{\Delta} = \frac{\partial V}{\partial S} = \frac{\partial}{\partial S}\left(K e^{-r\tau}(1 - N(d_-)) - S(1 - N(d_+))\right)$$

$$= \underbrace{\frac{\partial}{\partial S}\left(K e^{-r\tau} - S\right)}_{\text{I}} + \underbrace{\frac{\partial}{\partial S}\left(S N(d_+) - K e^{-r\tau} N(d_-)\right)}_{\text{I - Call}}$$

$$= -1 + N(d_+)$$

$$= \underline{N(d_+) - 1}$$

③

Call

$$\underset{\sim}{\Gamma} = \frac{\partial^2 V}{\partial S^2} = \frac{\partial \Delta}{\partial S} = \frac{\partial}{\partial S}\left(N(d_+)\right)$$

$$= \varphi(d_+)\frac{1}{\sigma\sqrt{\tau}}\cdot\frac{\cancel{K}}{S}\cdot\frac{1}{\cancel{K}} = \underline{\frac{\varphi(d_+)}{\sigma\sqrt{\tau}S}}$$

$=$

Put

$$\underset{\sim}{\Gamma} = \frac{\partial}{\partial S}\left(N(d_+)-1\right) = \frac{\partial}{\partial S}\left(N(d_+)\right) \underset{\text{Call}}{=} \underset{\sim}{\frac{\varphi(d_+)}{\sigma\sqrt{\tau}S}}$$

Call

$$\underline{\mathcal{V}} = \frac{\partial V}{\partial \sigma} = \frac{\partial}{\partial \sigma}\left(SN(d_+) - Ke^{-r\tau}N(d_-)\right)$$

$$= S\varphi(d_+)\frac{\partial}{\partial\sigma}\left(\frac{\log\left(\frac{S}{K}\right)+\left(r+\frac{1}{2}\sigma^2\right)\tau}{\sigma\sqrt{\tau}}\right)$$

$$- Ke^{-r\tau}\varphi(d_-)\frac{\partial}{\partial\sigma}\left(\frac{\log\left(\frac{S}{K}\right)+\left(r-\frac{1}{2}\sigma^2\right)\tau}{\sigma\sqrt{\tau}}\right)$$

$$= S\varphi(d_+)\left(\frac{\sigma\tau}{\sigma\sqrt{\tau}} - \frac{\log\left(\frac{S}{K}\right)+\left(r+\frac{1}{2}\sigma^2\right)\tau}{\sigma^2\sqrt{\tau}}\right)$$

$$-\underline{S\varphi(d_+)}\left(-\frac{\sigma\tau}{\sigma\sqrt{\tau}} - \frac{\log\left(\frac{S}{K}\right)+\left(r-\frac{1}{2}\sigma^2\right)\tau}{\sigma^2\sqrt{\tau}}\right)$$

Lemma

$\varphi(d_-)Ke^{-r\tau}$

$$= S\varphi(d_+)\left(\sqrt{\tau}+\sqrt{\tau} - \underbrace{\frac{1}{2\sigma^2\sqrt{\tau}}\sigma^2\tau - \frac{1}{2\sigma^2\sqrt{\tau}}\sigma^2\tau}_{\sqrt{\tau}}\right) = \underline{S\varphi(d_+)\sqrt{\tau}}$$

Put

$$\nu = \frac{\partial V}{\partial \sigma} = \frac{\partial}{\partial \sigma}\left(Ke^{-r\tau}(1-N(d_-)) - S(1-N(d_+))\right)$$

$$= \frac{\partial}{\partial \sigma}\left(-Ke^{-r\tau}N(d_-) + SN(d_+)\right)$$

$$\overset{\text{call}}{=} S\varphi(d_+)\sqrt{\tau}$$ - same as for the call

Call

not always the same convention

$$\Theta = -\frac{\partial V}{\partial \tau} = -\frac{\partial}{\partial \tau}\left(SN\left(\frac{\log(\frac{S}{K})+(r+\frac{1}{2}\sigma^2)\tau}{\sigma\sqrt{\tau}}\right) - Ke^{-r\tau}N\left(\frac{\log(\frac{S}{K})+(r-\frac{1}{2}\sigma^2)\tau}{\sigma\sqrt{\tau}}\right)\right)$$

$$= -S\varphi(d_+)\left(\frac{r+\frac{1}{2}\sigma^2}{\sigma\sqrt{\tau}} - \frac{1}{2}\frac{\log(\frac{S}{K})+(r+\frac{1}{2}\sigma^2)\tau}{\sigma\tau^{3/2}}\right)$$

$$+ Ke^{-r\tau}(-r)N(d_-)$$

$$+ \underbrace{Ke^{-r\tau}\varphi(d_-)}_{= S\varphi(d_+)}\left(\frac{r-\frac{1}{2}\sigma^2}{\sigma\sqrt{\tau}} - \frac{1}{2}\frac{\log(\frac{S}{K})+(r-\frac{1}{2}\sigma^2)\tau}{\sigma\tau^{3/2}}\right)$$

$$= -Kre^{-r\tau}N(d_-) - S\varphi(d_+)\left(\underbrace{\frac{1}{2}\frac{\sigma}{\sqrt{\tau}} + \frac{1}{2}\frac{\sigma}{\sqrt{\tau}} - \frac{1}{4}\frac{\sigma}{\sqrt{\tau}} - \frac{1}{4}\frac{\sigma}{\sqrt{\tau}}}_{= \frac{1}{2}\frac{\sigma}{\sqrt{\tau}}}\right)$$

$$= -Kre^{-r\tau}N(d_-) - \frac{S\sigma}{2\sqrt{\tau}}\varphi(d_+)$$

Put

$$\Theta = -\frac{\partial V}{\partial \tau} = -\frac{\partial}{\partial \tau}\left(Ke^{-r\tau}(1-N(d_-)) - S(1-N(d_+))\right)$$

$$= -\frac{\partial}{\partial \tau}\left(Ke^{-r\tau} - S\right) - \underbrace{\frac{\partial}{\partial \tau}\left(SN(d_+) - Ke^{-r\tau}N(d_-)\right)}_{= \text{Call}}$$

$$= -Ke^{-r\tau}(-r) - Kre^{-r\tau}N(d_-) - \frac{S\sigma}{2\sqrt{\tau}}\varphi(d_+)$$

$$= rKe^{-r\tau}(1-N(d_-)) - \frac{S\sigma}{2\sqrt{\tau}}\varphi(d_+)$$

Call

$$\rho = \frac{\partial V}{\partial r} = \frac{\partial}{\partial r}\left(SN\left(\frac{\log(\frac{S}{K}) + (r+\frac{1}{2}\sigma^2)\tau}{\sigma\sqrt{\tau}}\right) - Ke^{-r\tau}N\left(\frac{\log(\frac{S}{K}) + (r-\frac{1}{2}\sigma^2)\tau}{\sigma\sqrt{\tau}}\right)\right)$$

$$= S\varphi(d_+)\cdot\frac{1}{\sigma\sqrt{\tau}}\cdot\tau - Ke^{-r\tau}(-\tau)N(d_-)$$

$$- \underbrace{Ke^{-r\tau}\varphi(d_-)}_{= S\varphi(d_+)}\cdot\frac{1}{\sigma\sqrt{\tau}}\cdot\tau$$

$$= \tau Ke^{-r\tau}N(d_-)$$

Put

$$\underline{\rho} = \frac{\partial V}{\partial r} = \frac{\partial}{\partial r}\left(Ke^{-r\tau}(1-N(d_-)) - S(1-N(d_+))\right)$$

$$= \frac{\partial}{\partial r}\left(Ke^{-r\tau} - S\right) + \underbrace{\frac{\partial}{\partial r}\left(SN(d_+) - Ke^{-r\tau}N(d_-)\right)}_{\rho\text{ - Call}}$$

$$= Ke^{-r\tau}(-\tau) + \tau Ke^{-r\tau}N(d_-)$$

$$= \underline{-\tau Ke^{-r\tau}(1-N(d_-))}$$

Remarks:

1) $0 < \Delta_c < 1$

$$\Delta_c = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\frac{\log(\frac{S}{K}) + (r+\frac{1}{2}\sigma^2)\tau}{\sigma\sqrt{\tau}}} e^{-\frac{1}{2}y^2}\,dy \quad \xrightarrow{S\downarrow 0} 0, \quad \xrightarrow{S\uparrow\infty} 1$$

2) $-1 < \Delta_p < 0$ - shift

3) $\Gamma_c = \Gamma_p > 0$

4) $\nu_c = \nu_p > 0$

2) Consider

$$S_T = S_0 e^{(\mu - \frac{1}{2}\sigma^2)T + \sigma W_T} \quad (W_t) - P\text{-sBM}$$

$$S_T = S_0 e^{(r - \frac{1}{2}\sigma^2)T + \sigma B_T} \quad (B_t) - Q\text{-sBM}$$

$\mu = 0.16$, $\sigma = 0.35$, $S_0 = 38$, $K = 40$

$T = 3/12 = 1/4$, $r = 0.10$

a) Probability that call is exercised:

$$S_T = S_0 e^{\eta}, \; \eta \sim N\left((\mu - \tfrac{1}{2}\sigma^2)T, \sigma^2 T\right)$$

$$\eta \sim \underbrace{N\left((0.16 - \tfrac{1}{2}\cdot 0.35^2)\cdot 0.25, 0.35^2\cdot 0.25\right)}_{\parallel}$$

$$N(0.0247, 0.0306)$$

$$\underline{P(S_T > 40)} = P(38e^{\eta} > 40) = P(\eta > \underbrace{\log(\tfrac{40}{38})}_{\parallel\ 0.0513})$$

$$= P\left(\frac{\eta - 0.0247}{\sqrt{0.0306}} > \frac{0.0513 - 0.0247}{\sqrt{0.0306}}\right)$$

$$= 1 - N(0.152) = 1 - 0.5604$$

$$= \underline{0.4396}$$

Value of the call

$$d_+ = \frac{\log\left(\frac{38}{40}\right) + \left(0.1 + \frac{1}{2}0.35^2\right)\cdot 0.25}{0.35\sqrt{0.25}} = -0.06275$$

$$d_- = -0.06275 - 0.35\sqrt{0.25} = -0.23775$$

$$\Downarrow$$

$$\underset{\sim}{c} = 38N(-0.06275) - 40e^{-0.1\cdot 0.25}N(-0.23775)$$

$$= \underline{2.209}$$

b) Put

$$\underline{P(S_T < 40)} = 1 - P(S_T \geq 40)$$

$$= 1 - 0.4396 = \underline{0.5604}$$

$$\underset{\sim}{P} = 40e^{-0.1\cdot 0.25}N(\underbrace{+0.23775}_{-d_-}) - 38N(\underbrace{0.0675}_{-d_+})$$

$$= \underline{3.22}$$

3) Consider $f(S_T) = \max(S_T, k)$

$t = 0$

$$\underline{V_0} = e^{-rT} E_Q(\max(S_T, k))$$

$$= e^{-rT} E_Q((S_T - k)_+) + e^{-rT} E_Q(k)$$

$$= S_0 N(d_+) - k e^{-rT} N(d_-) + e^{-rT} k$$

$$= S_0 N(d_+) + k e^{-rT}(1 - N(d_-))$$

$$\underline{= S_0 N(d_+) + \underbrace{1}_{S_0^0} \cdot k e^{-rT} N(-d_-)}$$

$$H_0 = \frac{\partial V_0}{\partial S_0} = \frac{\partial C}{\partial S_0} + \underbrace{\frac{\partial}{\partial S_0}(e^{-rT} K)}_{=0} = \underline{N(d_+)}$$

$$\underline{H_0^0} = V_0 - H_0 S_0 = S_0 N(d_+) + 1 k e^{-rT} N(-d_-) - N(d_+) S_0$$

$$= \underline{k e^{-rT}(1 - \overbrace{N(d_-))}^{N(-d_-)}}$$

4.1) Is X defined by

$$X_t = 2(W_{1+\frac{t}{4}} - W_1) \text{ a BM ?}$$

a) X - continuous as being a composition of two continuous functions, $W_\cdot$ and $1+\frac{\cdot}{4}$

b) $t > s$

$\Downarrow$

$$X_t - X_s = 2(W_{1+\frac{t}{4}} - W_1) - 2(W_{1+\frac{s}{4}} - W_1)$$

$$= 2(W_{1+\frac{t}{4}} - W_{1+\frac{s}{4}})$$

where X_s is $\mathcal{F}_{1+\frac{s}{4}}$-measurable

and $X_t - X_s$ is independent of $\mathcal{F}_{1+\frac{s}{4}}$

c) $W_u \sim$ Gaussian for all u

and also for increments

Consider

$W_v - W_u$ with $v = 1 + \frac{t}{4}$, $u = 1 + \frac{s}{4}$,

$W_v - W_u \sim N(0, v - u)$

$$1 + \frac{t}{4} - 1 - \frac{s}{4} = \frac{1}{4}(t-s)$$

Hence,

$$Var(2(W_v - W_u)) = 4 Var(W_v - W_u)$$

$$= 4\left(\frac{1}{4}(t-s)\right) = t - s$$

Hence, (X_t) is a sBM w.r.t. $\left(\mathcal{F}_{1+\frac{t}{4}}\right) = \left(\tilde{\mathcal{F}}_t\right)$

4.2) Consider (Y_t) defined by

$$Y_t = \sqrt{t}\, W_1 .$$

Clearly $Y_t \sim N(0,t)$, but

$$Y_4 = 2W_1 = 2Y_1$$

$$\Downarrow$$

$$Y_4 - Y_1 = 2Y_1 - Y_1 = Y_1 = Y_1 - Y_0,$$

i.e. the increments are not independent.

4.3) Consider (Z_t) defined by

$Z_t = W_{2t} - W_t$. Hence,

$$Z_t - Z_s = W_{2t} - W_t - \overbrace{(W_{2s} - W_s)}^{\mathcal{F}^W_{2s}\text{-measurable}}$$

$$= (W_{2t} - W_{2s}) - (W_t - W_s), \text{ consider}$$

$W_{2t} - W_{2s}$ - independent of $\mathcal{F}_{2s}$

$W_t - W_s$ - measurable w.r.t. $\mathcal{F}_{2s}$

Hence,

$Z_t - Z_s$ cannot be independent of $\mathcal{F}_{2s}$.

Assume that it is. Then

$$Z_t - Z_s - (W_{2t} - W_{2s}) = -(W_t - W_s)$$

is independent of $\mathcal{F}_{2s}$!