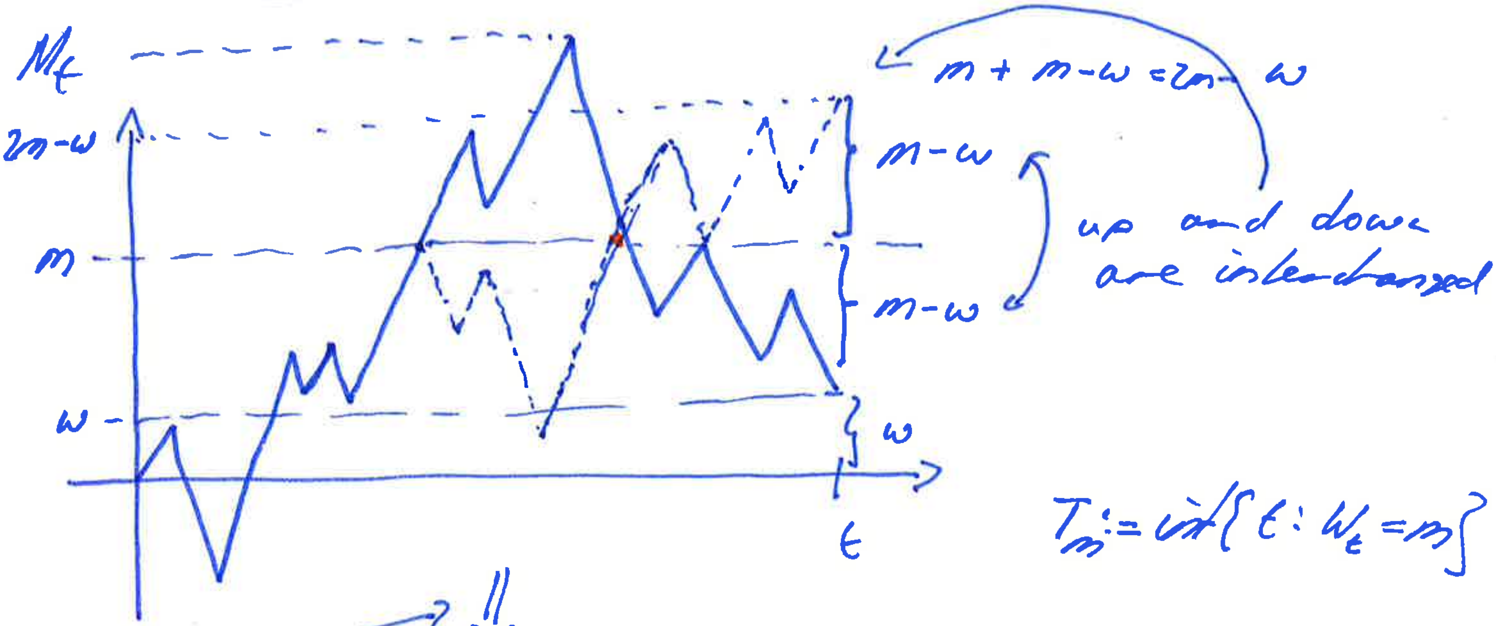
Exercises; extra material W7



$T_m := \inf\{t : W_t = m\}$

$\Downarrow$

$$P(T_m \le t, W_t \le w) = P(W_t \ge 2m-w)$$

Heuristic argument: Fix a positive level m and a positive time t. In order to "count" the Brownian motion paths that reach level m at or before time t note that there are two types of such paths: those that reach level m prior to t but at time t are at some level w below m, and those that exceed level m at time t (there are also B.M. paths that are exactly at level m at time t, but the probability of these paths is vanishing, i.e. we may ignore this possibility). For each Brownian motion path that reaches level m prior to time t but is at level w below m at time t, there is a "reflected path" that is at level $2m-w$ at time t, where this reflected path is constructed by switching the up and down moves of the B.M. from time $T_m = \inf\{t : W_t = m\}$ onward. Of course, the probability that a Brownian motion path ends exactly at w or at exactly $2m-w$ is zero. Hence, in order to have nonzero probabilities, consider the paths that reach level m prior to time t and are at or below level w at time t, and consider their reflections, which are at or above $2m-w$ at time t.

$$\Downarrow$$

$$P(T_m \le t, W_t \le w) = P(W_t \ge 2m-w)$$

$$w \le m, \ m > 0$$

Clearly $\{T_m \le t\}$ coincides with $\{M_t = \sup_{s \le t} W_s \ge m\}$ for $m > 0 \Rightarrow$

$$P(M_t \ge m, W_t \le w) = P(W_t \ge 2m-w).$$

More formally:

Let $S_m = \inf\{t : W_t \geq m\}$ - $(\mathcal{F}_t)$-stopping time

and note that

$$\{S_m \leq t\} = \{M_t \geq m\}$$

where by the continuity of Brownian motion paths

$$S_m = T_m = \inf\{t : W_t = m\}$$

and $W_{T_m} = m$.

Thus,

$$P(M_t \geq m, W_t \leq w) = P(T_m \leq t, W_t \leq w)$$

on $\{T_m \leq t\}$ is $W_{T_m} = m$

$$= P(T_m \leq t, W_t - W_{T_m} \leq w - m)$$

$$= E(\mathbb{1}_{\{W_t - W_{T_m} \leq w-m\}} \mathbb{1}_{\{T_m \leq t\}})$$

$$= E(E(\mathbb{1}_{\{W_t - W_{T_m} \leq w-m\}} \underbrace{\mathbb{1}_{\{T_m \leq t\}}}_{\text{measurable}} | T_m))$$

$$= E(\mathbb{1}_{\{T_m \leq t\}} E(\mathbb{1}_{\{W_t - W_{T_m} \leq w-m\}} | T_m))$$

$$= E(\mathbb{1}_{\{T_m \leq t\}} \varphi(T_m))$$

Consider

$$\varphi(T_m) = E\left(\mathbb{1}_{\{W_t - W_{T_m} \le w-m\}} \mid T_m\right)$$

with

$(\tilde{W}_s) = (W_{T_m+s} - W_{T_m})$. $(\tilde{W}_s)$ is a Brownian motion independent of $\mathcal{F}_{T_m}$.

Note $\tilde{W}_{t-T_m} = W_{T_m+(t-T_m)} - W_{T_m} = W_t - W_{T_m}$

$$\varphi(x) = E\left(\mathbb{1}_{\{\tilde{W}_{t-x} \le w-m\}}\right).$$

Since $(-\tilde{W}_t)$ is also a BM we have

$$\begin{aligned}\varphi(x) &= E\left(\mathbb{1}_{\{\tilde{W}_{t-x} \le w-m\}}\right)\\ &= E\left(\mathbb{1}_{\{-\tilde{W}_{t-x} \le w-m\}}\right)\\ &= E\left(\mathbb{1}_{\{\tilde{W}_{t-x} \ge m-w\}}\right) = \tilde{\varphi}(x)\end{aligned}$$

Hence,

$$\varphi(T_m) = \tilde{\varphi}(T_m) = E\left(\mathbb{1}_{\{W_t - W_{T_m} \ge m-w\}} \mid T_m\right)$$

Hence,

$$E(\mathbb{1}_{T_m \le t}\,\varphi(T_m)) = E(\mathbb{1}_{\{T_m \le t\}}\,\tilde{\varphi}(T_m))$$

$$= E(\mathbb{1}_{\{T_m \le t\}}\,E(\mathbb{1}_{\{W_t - W_{T_m} \ge m-w\}} \mid T_m))$$

$$= E(E(\mathbb{1}_{\{T_m \le t\}}\,\mathbb{1}_{\{W_t - W_{T_m} \ge m-w\}} \mid T_m))$$

$$= E(\mathbb{1}_{\{T_m \le t\}}\,\mathbb{1}_{\{W_t - W_{T_m} \ge m-w\}})$$

$$= P(T_m \le t,\ W_t - \overbrace{W_{T_m}}^{= m \text{ on } \{T_m \le t\}} \ge m-w)$$

$$= P(T_m \le t,\ W_t \ge 2m-w)$$

$\Downarrow$ $\{T_m \le t\} = \{M_t \ge m\}$

$$\underbrace{P(M_t \ge m,\ W_t \le w)}_{\text{starting point}} = P(W_t \ge 2m-w,\ M_t \ge m)$$

$$= P(W_t \ge 2m-w)$$

$w \le m$

$\Updownarrow$

$0 \le m-w \Rightarrow m \le m + (m-w) = 2m-w$

Hence, $W_t \ge 2m-w \Rightarrow M_t \ge W_t \ge m$

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Thus, $\{W_t \geq 2m - w\} \Rightarrow \{M_t \geq m\}$

$$\{M_t \geq m\} = \{T_m \leq t\}$$

$$\Downarrow$$

$$P(T_m \leq t, W_t \leq w) = P(W_t \geq 2m - w)$$

Exercise 2: Distribution of (M_t, W_t)

From Ex. 1

$$P(M_t \geq m, W_t \leq w) = P(W_t \geq 2m - w), \quad m > 0, \ w \leq m.$$

$$P(M_t \geq m, W_t \leq w) = \underbrace{\int_m^\infty \int_{-\infty}^w f(x,y)\,dy\,dx}_{\text{differentiable in } m}$$

$$P(W_t \geq 2m - w) = \frac{1}{\sqrt{2\pi t}} \int_{2m-w}^\infty e^{-\frac{z^2}{2t}}\,dz = \underbrace{1 - \frac{1}{\sqrt{2\pi t}} \int_{-\infty}^{2m-w} e^{-\frac{z^2}{2t}}\,dz}_{\text{differentiable in } m}$$

$$\Downarrow \frac{\partial}{\partial m}$$

$$-\int_{-\infty}^w f(m,y)\,dy = -\frac{1}{\sqrt{2\pi t}} e^{-\frac{(2m-w)^2}{2t}} \cdot 2 = -\frac{2}{\sqrt{2\pi t}} e^{-\frac{(2m-w)^2}{2t}}$$

Clearly $-\frac{2}{\sqrt{2\pi t}} e^{-\frac{(2m-w)^2}{2t}}$ is

differentiable in w and so is the l.h.s.

$$\frac{d}{dw}\left(-\int_{-\infty}^{w} f(m,y)\,dy\right) = \frac{d}{dw}\left(-\frac{2}{\sqrt{2\pi t}} e^{-\frac{(2m-w)^2}{2t}}\right)$$

$$\parallel \qquad\qquad\qquad\qquad \parallel$$

$$-f(m,w) \qquad = \underbrace{-\frac{2}{\sqrt{2\pi t}} e^{-\frac{(2m-w)^2}{2t}}\left(-\frac{2(2m-w)}{2t}\right)(-1)}_{-\frac{2(2m-w)}{t\sqrt{2\pi t}} e^{-\frac{(2m-w)^2}{2t}}}$$

Hence, for $m>0$, $w\le m$

$$\underline{f(m,w) = \frac{2(2m-w)}{t\sqrt{2\pi t}} e^{-\frac{(2m-w)^2}{2t}}}$$

Furthermore, note for the cdf

$$P(A\cap B) = P(A\setminus(A\cap B^c))$$
$$= P(A) - P(A\cap B^c), \quad A=\{W_t\le w\}, \ B=\{M_t\le m\}$$

1) $m>0$, $w\le m$

$$P(M_t\le m, W_t\le w) = P(W_t\le w) - \underbrace{P(M_t\ge m, W_t\le w)}$$
$$= P(W_t\le w) - P(W_t\ge 2m-w)$$

Reflexion principle

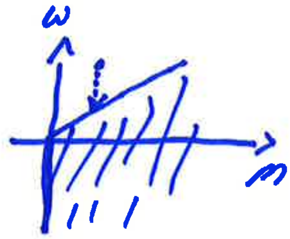
Hence, for $m>0$, $w \le m$

$$P(M_t \le m, W_t \le w) = P(W_t \le w) - P(W_t \ge 2m - w)$$

$$= N\left(\frac{w}{\sqrt{t}}\right) - \left(1 - N\left(\frac{2m-w}{\sqrt{t}}\right)\right)$$

$$= N\left(\frac{w}{\sqrt{t}}\right) - N\left(\frac{w-2m}{\sqrt{t}}\right)$$

2) $0 < m \le w$, $M_t \ge W_t$ - always



$$P(M_t \le m, W_t \le w) = P(M_t \le m, W_t \le m)$$

as above $$= N\left(\frac{m}{\sqrt{t}}\right) - N\left(\frac{m-2m}{\sqrt{t}}\right)$$

$$= N\left(\frac{m}{\sqrt{t}}\right) - N\left(-\frac{m}{\sqrt{t}}\right)$$

3) While for $m \le 0$, $M_t \ge M_0 = 0$ so that

$$P(M_t \le m, W_t \le w) = 0$$

Remarks:

- From 2) $\left(\{M_t \le m\} \Rightarrow \{W_t \le m\} \right)$

$$P(M_t \le m) = N\left(\frac{m}{\sqrt{t}}\right) - N\left(-\frac{m}{\sqrt{t}}\right) = P(W_t \le m) - P(W_t \le -m)$$

$$= P(|W_t| \le m) \qquad m > 0$$

Furthermore,

$$\frac{\partial}{\partial m}\left(P(M_t \le m)\right) = \frac{1}{\sqrt{2\pi t}} \exp\left(-\frac{1}{2}\frac{m^2}{t}\right) + \frac{1}{\sqrt{2\pi t}} \exp\left(-\frac{1}{2}\frac{m^2}{t}\right)$$

$$= \frac{2}{\sqrt{2\pi t}} \exp\left(-\frac{1}{2}\frac{m^2}{t}\right), \quad m > 0$$

- $$f_{(M_t, W_t)}(m, w) = \frac{2(2m-w)}{t\sqrt{2\pi t}} e^{-\frac{(2m-w)^2}{2t}} \mathbb{1}_{\{m>0\}} \mathbb{1}_{\{w \le m\}}$$

Density of $(M_t, M_t - W_t) = (M_t, Y_t)$

$$\begin{matrix} u = m \\ v = m - w \end{matrix} \iff \begin{matrix} m = u \\ w = u - v \end{matrix} \iff \begin{pmatrix} m \\ w \end{pmatrix} \overset{\varphi}{=} \begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}$$

$$|\det J_\varphi| = \left|\det \begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix}\right| = 1$$

Rewrite,

$$f_{(M_t,W_t)}(m,w) = \sqrt{\frac{2}{\pi t}}\,\frac{(2m-w)}{t}\,e^{-\frac{(2m-w)^2}{2t}}\,\mathbb{1}_{\{m\ge 0\}}\mathbb{1}_{\{w\le m\}}$$

Thus,

$$f_{(M_t,Y_t)}(u,v) = \sqrt{\frac{2}{\pi t}}\,\frac{2u-(u-v)}{t}\,e^{-\frac{(2u-(u-v))^2}{2t}}\,\mathbb{1}_{\{u\ge 0\}}\mathbb{1}_{\{0\le v\}}$$

$\{u-v\le u\}$

$\{0\le v\}$

$$= \sqrt{\frac{2}{\pi t}}\,\frac{(u+v)}{t}\exp\left(-\frac{(u+v)^2}{2t}\right)\mathbb{1}_{\{u\ge 0\}}\mathbb{1}_{\{v\ge 0\}}$$

$\Downarrow$

(M_t,Y_t) - exchangeable

$\Downarrow$

M_t and Y_t

Exercise 3

(W_t) - BM

$\hat{W} = (\alpha t + W_t)_{t\in[0,T]}$, $(\hat{W}_t)_{t\in[0,T]}$ - BM with drift α

$\hat{M}_T = \sup_{0\le t\le T} \hat{W}_t$

Define the exponential martingale, by

$$\hat{Z}_t = e^{-\alpha W_t - \frac{1}{2}\alpha^2 t} \quad - (\hat{Z}_t)\text{- martingale}$$

$$= e^{-\alpha(\hat{W}_t - \alpha t) - \frac{1}{2}\alpha^2 t}$$

$$= e^{-\alpha \hat{W}_t + \frac{1}{2}\alpha^2 t} \qquad 0\le t\le T.$$

Use $\hat{Z}_T$ in order to define a new probability measure P by

$$P(A) = \int_A \hat{Z}_T \, dQ, \qquad \forall A \in \mathcal{F}, \text{ i.e.}$$

$$\frac{dP}{dQ} = \hat{Z}_T$$

$\Downarrow$ - Girsanov

$(\hat{W}_t)_{t\in[0,T]}$ - is a standard BM w.r.t P

From Ex. 2: Under P $(\hat{M}_T, \hat{W}_T)$ has a density

$$\hat{f}(m,\omega) = \frac{2(2m-\omega)}{T\sqrt{2\pi T}} e^{-\frac{(2m-\omega)^2}{2T}} \qquad \begin{matrix}\omega \le m \\ m > 0\end{matrix}$$

Hence, for $\omega \le m$, $m > 0$

$$Q(\hat{M}_T \le m, \hat{W}_T \le \omega) = E_Q(\mathbb{1}_{\{\hat{M}_T \le m,\ \hat{W}_T \le \omega\}})$$

$$= \int_\Omega \mathbb{1}_{\{\hat{M}_T(\omega) \le m,\ \hat{W}_T(\omega) \le \omega\}} dQ(\omega)$$

(heuristically $dQ = \frac{dQ}{dP} dP$, $\frac{dQ}{dP} = \frac{1}{\hat{Z}}$)

$$= \int_\Omega \mathbb{1}_{\{\hat{M}_T(\omega) \le m,\ \hat{W}_T(\omega) \le \omega\}} \frac{1}{\hat{Z}_T(\omega)} dP(\omega)$$

$$= E_P\left(\frac{1}{\hat{Z}_T} \mathbb{1}_{\{\hat{M}_T \le m,\ \hat{W}_T \le \omega\}}\right)$$

$$= E_P\left(e^{\alpha \hat{W}_T - \frac{\alpha^2}{2} T} \mathbb{1}_{\{\hat{M}_T \le m,\ \hat{W}_T \le \omega\}}\right)$$

$$= \int_{-\infty}^{\omega} \int_0^m e^{\alpha y - \frac{1}{2}\alpha^2 T} \hat{f}(x,y)\, dx\, dy$$

$$\Downarrow$$

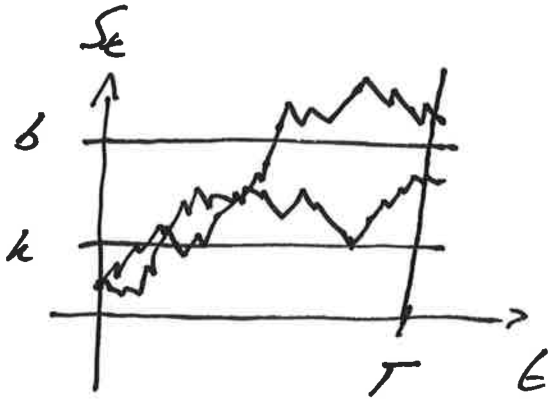
$(\hat{M}_T, \hat{W}_T)$ under Q has density

$$\underline{f(m,\omega) = e^{\alpha \omega - \frac{1}{2}\alpha^2 T} \hat{f}(m,\omega)} \quad \text{for } \omega \le m,\ m > 0$$

Ex. 4)

$$h = (S_T - k)_+ \mathbb{1}_{\{S_t < b, \forall t \in [0,T]\}}$$

where $S_0 < k < b$



$$\infty > E_Q(S_T^2) \geq E_Q((S_T - k)_+^2) \geq E_Q((\underbrace{(S_T - k)_+ \mathbb{1}_{\{\dots\}}}_{= h})^2)$$

$\Downarrow$

h - square integrable

and $\mathcal{F}_T$ - measurable,

furthermore, h - non-negative

$\Downarrow$

h - claim

Hence,

$$V_0 = e^{-rT} E_Q\left((S_T - k)_+ \mathbb{1}_{\{S_t < b,\ \forall t \in [0,T]\}} \right)$$

$$\parallel$$

$$e^{-rT} E_Q\left(\left(S_0 e^{(r-\frac{1}{2}\sigma^2)T + \sigma W_T} - k\right)_+ \mathbb{1}_{\{S_0 e^{(r-\frac{1}{2}\sigma^2)t + \sigma W_t} < b,\ \forall t \in [0,T]\}} \right)$$

$$\parallel$$

$$e^{-rT} E_Q\left(\left(S_0 e^{\sigma(\frac{1}{\sigma}(r-\frac{1}{2}\sigma^2)T + W_T)} - k\right)_+ \mathbb{1}_{\{S_0 e^{\sigma(\frac{1}{\sigma}(r-\frac{1}{2}\sigma^2)t + W_t)} < b,\ \forall t \in [0,T]\}} \right)$$

$$\parallel$$

$$e^{-rT} E_Q\left(\left(S_0 e^{\sigma \hat{W}_T} - k\right)_+ \mathbb{1}_{\{S_0 e^{\sigma \hat{W}_t} < b,\ \forall t \in [0,T]\}} \right)$$

for $\hat{W}_t = \alpha t + W_t$, $\alpha = \frac{1}{\sigma}(r - \frac{1}{2}\sigma^2)$

$$\parallel$$

from the call

$$e^{-rT} E_Q\left(\left(S_0 e^{\sigma \hat{W}_T} - k\right) \mathbb{1}_{\{k < S_0 e^{\sigma \hat{W}_T} < b\}} \mathbb{1}_{\{S_0 e^{\sigma \hat{W}_t} < b,\ \forall t \in [0,T]\}} \right.$$

Hence, $V_0 =$

$$e^{-rT} E_Q\left((S_0 e^{\sigma \hat{W}_T} - k) \mathbb{1}_{\{k < S_0 e^{\sigma \hat{W}_T} < b\}} \mathbb{1}_{\{S_0 e^{\sigma \hat{W}_t} < b,\ \forall t \in [0,T]\}} \right)$$

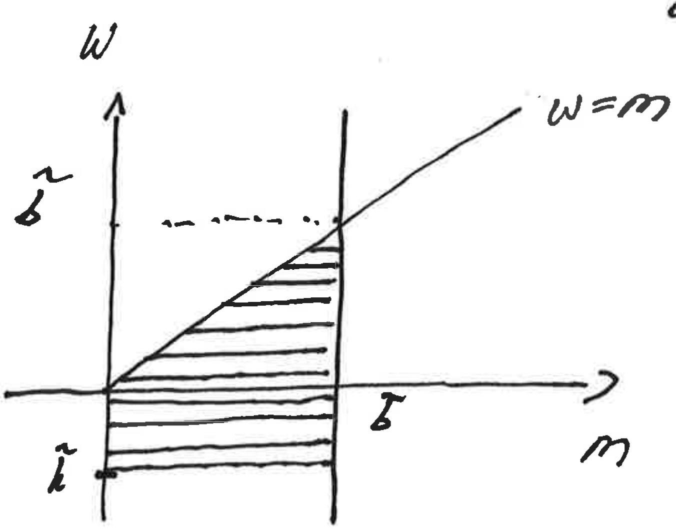
$$\|$$

$$e^{-rT} E_Q\Bigg((S_0 e^{\sigma \hat{W}_T} - k) \mathbb{1}_{\left\{ \underbrace{\frac{\log\left(\frac{k}{S_0}\right)}{\sigma}}_{\tilde{k}} < \hat{W}_T < \underbrace{\frac{\log\left(\frac{b}{S_0}\right)}{\sigma}}_{\tilde{b}} \right\}} \mathbb{1}_{\left\{ \hat{M}_T < \underbrace{\frac{\log\left(\frac{b}{S_0}\right)}{\sigma}}_{\tilde{b}} \right\}} \Bigg)$$

$$\hat{M}_T = \sup_{t \in [0,T]} \hat{W}_t$$

$$\|$$

$$e^{-rT} \iint (S_0 e^{\sigma w} - k) \mathbb{1}_{\{\tilde{k} < w < \tilde{b}\}} \mathbb{1}_{\{m < \tilde{b}\}} \underbrace{\mathbb{1}_{\{m > 0\}} \mathbb{1}_{\{w \le m\}} \frac{2(2m-w)}{T\sqrt{2\pi T}} e^{\alpha w - \frac{1}{2}\alpha^2 T - \frac{1}{2T}(2m-w)^2}}_{\text{density}} \, dm\, dw$$



$$= e^{-rT}\int_{\tilde{k}}^{\tilde{b}} (S_0 e^{\sigma w} - k)\,\frac{e^{\alpha w - \frac{1}{2}\alpha^2 T}}{\sqrt{2\pi T}}\;\int_{w^+}^{\tilde{b}} \frac{2(2m-w)}{T}\, e^{-\frac{1}{2T}(2m-w)^2}\, dm\, dw$$

$$m = w$$

$$z = (2m-w)^2$$

$$\left|\frac{dz}{dm}\right|^{-1} = \left(2(2m-w)\cdot 2\right)^{-1}$$

$$= \int_{z(w^+)}^{z(\tilde{b})} \frac{1}{2T}\, e^{-\frac{1}{2T}z}\, dz$$

$$= -\frac{2T}{2T}\, e^{-\frac{1}{2T}z}\Big|_{z(w^+)}^{z(\tilde{b})}$$

$$= e^{-rT}\int_{\tilde{k}}^{\tilde{b}} (S_0 e^{\sigma w} - k)\,\frac{e^{\alpha w - \frac{1}{2}\alpha^2 T}}{\sqrt{2\pi T}}\left(-e^{-\frac{1}{2T}(2m-w)^2}\Big|_{w^+}^{\tilde{b}}\right) dw$$

(note that $(0-w)^2 = w^2$ and $(2w-w)^2 = w^2$

$$= \int_{\tilde{k}}^{\tilde{b}} (S_0 e^{\sigma w} - k)\,\frac{e^{-rT + \alpha w - \frac{1}{2}\alpha^2 T - \frac{1}{2T}w^2}}{\sqrt{2\pi T}}\, dw$$

$$- \int_{\tilde{k}}^{\tilde{b}} (S_0 e^{\sigma w} - k)\,\frac{e^{-rT + \alpha w - \frac{1}{2}\alpha^2 T - \frac{1}{2T}(2\tilde{b}-w)^2}}{\sqrt{2\pi T}}\, dw$$

$$= S_0 \underbrace{\left(\frac{1}{\sqrt{2\pi T}} \int_{\tilde{k}}^{\tilde{b}} e^{\sigma\omega - rT + \alpha\omega - \frac{1}{2}\alpha^2 T - \frac{1}{2T}\omega^2} \, d\omega \right)}_{I_1}$$

$$- K \underbrace{\left(\frac{1}{\sqrt{2\pi T}} \int_{\tilde{k}}^{\tilde{b}} e^{-rT + \alpha\omega - \frac{1}{2}\alpha^2 T - \frac{1}{2T}\omega^2} \, d\omega \right)}_{I_2}$$

$$- S_0 \left(\frac{1}{\sqrt{2\pi T}} \int_{\tilde{k}}^{\tilde{b}} e^{\sigma\omega - rT + \alpha\omega - \frac{1}{2}\alpha^2 T \underbrace{- \frac{1}{2T}(2\tilde{b} - \omega)^2}_{=}} \, d\omega \right.$$

$$- \frac{1}{2T}\left(4\tilde{b}^2 - 4\tilde{b}\omega + \omega^2\right)$$

$$= -\frac{2\tilde{b}^2}{T} + \frac{2\tilde{b}\omega}{T} - \frac{\omega^2}{2T}$$

$$\underbrace{\frac{1}{\sqrt{2\pi T}} \int_{\tilde{k}}^{\tilde{b}} e^{\sigma\omega - rT + \alpha\omega - \frac{1}{2}\alpha^2 T - \frac{2\tilde{b}^2}{T} + \frac{2\tilde{b}\omega}{T} - \frac{\omega^2}{2T}} \, d\omega}_{I_3}$$

$$+ K \underbrace{\left(\frac{1}{\sqrt{2\pi T}} \int_{\tilde{k}}^{\tilde{b}} e^{-rT + \alpha\omega - \frac{1}{2}\alpha^2 T - \frac{2\tilde{b}^2}{T} + \frac{2\tilde{b}\omega}{T} - \frac{\omega^2}{2T}} \, d\omega \right)}_{I_4}$$

$$= S_0 I_1 - K I_2 - S_0 I_3 + K I_4$$

Observe that each of these integrals are of the form

$$\frac{1}{\sqrt{2\pi T}} \int_{\tilde a}^{\tilde b} e^{\beta + \gamma w - \frac{w^2}{2T}}\, dw$$

– compl. the square

$$= \frac{1}{\sqrt{2\pi T}} \int_{\tilde a}^{\tilde b} e^{\beta + \frac{1}{2}\gamma^2 T - \frac{1}{2T}(\gamma^2 T^2 - 2\gamma w T + w^2)}\, dw$$

$$= \frac{e^{\beta + \frac{1}{2}\gamma^2 T}}{\sqrt{2\pi T}} \int_{\tilde a}^{\tilde b} e^{-\frac{1}{2T}(w - \gamma T)^2}\, dw$$

$$y = \frac{w - \gamma T}{\sqrt{T}}$$

$$\Updownarrow$$

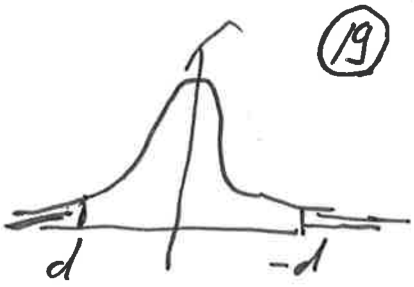
$$(w = \sqrt{T}\, y + \gamma T)$$

$$\left|\frac{dy}{dw}\right|^{-1} = \sqrt{T}$$

$$= \frac{e^{\beta + \frac{1}{2}\gamma^2 T}\, \cancel{\sqrt{T}}}{\sqrt{2\pi}\, \cancel{\sqrt{T}}} \int_{\frac{\tilde a - \gamma T}{\sqrt{T}}}^{\frac{\tilde b - \gamma T}{\sqrt{T}}} e^{-\frac{1}{2}y^2}\, dy$$

$$=$$

$$= e^{\frac{\nu^2 T}{2}+\beta}\left(N\left(\frac{\tilde{b}-\nu T}{\sqrt{T}}\right)-N\left(\frac{\tilde{h}-\nu T}{\sqrt{T}}\right)\right)$$



$$= e^{\frac{\nu^2 T}{2}+\beta}\left[1-N\left(\frac{-\tilde{b}+\nu T}{\sqrt{T}}\right)-\left(1-N\left(\frac{-\tilde{h}+\nu T}{\sqrt{T}}\right)\right)\right]$$

$$= e^{\frac{1}{2}\nu^2 T+\beta}\left[N\left(\frac{-\tilde{h}+\nu T}{\sqrt{T}}\right)-N\left(\frac{-\tilde{b}+\nu T}{\sqrt{T}}\right)\right]$$

$$= e^{\frac{1}{2}\nu^2 T+\beta}\left(N\left(\frac{-\log\left(\frac{h}{S_0}\right)+\nu\sigma T}{\sigma\sqrt{T}}\right)-N\left(\frac{-\log\left(\frac{b}{S_0}\right)+\nu\sigma T}{\sigma\sqrt{T}}\right)\right)$$

$$= e^{\frac{\nu^2 T}{2}+\beta}\left(N\left(\frac{\log\left(\frac{S_0}{h}\right)+\nu\sigma T}{\sigma\sqrt{T}}\right)-N\left(\frac{\log\left(\frac{S_0}{b}\right)+\nu\sigma T}{\sigma\sqrt{T}}\right)\right)$$

Introduce $\delta_\pm(s) = \dfrac{\log(s)+(r\pm\frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}$

Hence,

$$\frac{1}{\sqrt{2\pi T}}\int_{\tilde{h}}^{\tilde{b}} e^{\beta+\nu w-\frac{w^2}{2T}}\,dw = e^{\frac{1}{2}\nu^2 T+\beta}\left(N\left(\frac{\log\left(\frac{S_0}{h}\right)+\nu\sigma T}{\sigma\sqrt{T}}\right)-N\left(\frac{\log\left(\frac{S_0}{b}\right)+\nu\sigma T}{\sigma\sqrt{T}}\right)\right)$$

$$\left(e^{\frac{1}{2}\gamma^2 T+\beta} \quad \text{and} \quad \gamma\sigma T \quad \text{have to be computed}\right)$$

$$I_1 : \beta = -rT - \frac{1}{2}\alpha^2 T, \quad \gamma = \alpha + \sigma, \quad \text{i.e.}$$

$$\frac{1}{2}\gamma^2 T + \beta = \frac{1}{2}\overbrace{(\alpha^2 + 2\alpha\sigma + \sigma^2)}^{\gamma^2}T \overbrace{- rT - \frac{1}{2}\alpha^2 T}^{+\beta}$$

$$= \cancel{\frac{1}{2}\alpha^2 T} + \alpha\sigma T + \frac{1}{2}\sigma^2 T - rT - \cancel{\frac{1}{2}\alpha^2 T}$$

$$= \frac{(r - \frac{1}{2}\sigma^2)}{\sigma}\sigma T + \frac{1}{2}\sigma^2 T - rT$$

$$= \cancel{rT} - \cancel{\frac{1}{2}\sigma^2 T} + \cancel{\frac{1}{2}\sigma^2 T} - \cancel{rT} = 0$$

and

$$\gamma\sigma = (\alpha + \sigma)\sigma = \left(\overbrace{\frac{r - \frac{1}{2}\sigma^2}{\sigma}}^{\alpha} + \sigma\right)\sigma$$

$$= r \underbrace{- \frac{1}{2}\sigma^2 + \sigma^2} = r + \frac{1}{2}\sigma^2$$

$$\Downarrow \quad \text{Since } e^0 = 1$$

$$I_1 = N\left(\frac{\log(\frac{S_0}{k}) + (r + \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}\right) - N\left(\frac{\log(\frac{S_0}{k}) + (r + \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}\right)$$

$\left(e^{\frac{1}{2}\gamma^2 T + \beta}\right.$ and $\gamma\sigma T$ have to be componted$\left.\right)$

For I_2 : $\beta = -rT - \frac{1}{2}\alpha^2 T$, $\quad \gamma = \alpha$

$$\Downarrow$$

$$\frac{1}{2}\gamma^2 T + \beta = \frac{1}{2}\alpha^2 T \overbrace{-rT - \frac{1}{2}\alpha^2 T}^{+\beta} = -rT$$

and

$$\gamma\sigma = \sigma\alpha = \sigma\left(\frac{r - \frac{1}{2}\sigma^2}{\sigma}\right) = r - \frac{1}{2}\sigma^2$$

$$\Downarrow$$

$$I_2 = e^{-rT}\left(N\left(\frac{\log\left(\frac{S_0}{K}\right) + \left(r - \frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}}\right) - N\left(\frac{\log\left(\frac{S_0}{b}\right) + \left(r - \frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}}\right)\right)$$

$$= e^{-rT}\left(N\left(\delta_-\left(\frac{S_0}{k}\right)\right) - N\left(\delta_-\left(\frac{S_0}{b}\right)\right)\right).$$

$\left(e^{\frac{1}{2}\gamma^2 T+\beta}\right.$ and $\gamma\sigma T$ have to be computed)

For I_3, $\beta = -rT - \frac{1}{2}\alpha^2 T - \frac{2\tilde{b}^2}{T}$, $\gamma = \left(\sigma + \alpha + \frac{2\tilde{b}}{T}\right)$

so that

$$\frac{1}{2}\gamma^2 T + \beta = \frac{1}{2}\overbrace{\left(\sigma^2+\alpha^2+\frac{4\tilde{b}^2}{T^2}+2\sigma\alpha+\frac{4\tilde{b}\sigma}{T}+\frac{4\tilde{b}\alpha}{T}\right)}^{\gamma^2}T$$

$$-rT - \frac{1}{2}\alpha^2 T - \frac{2\tilde{b}^2}{T}$$

$$= \frac{1}{2}\sigma^2 T + \cancel{\frac{1}{2}\alpha^2 T} + \cancel{\frac{2\tilde{b}^2}{T}} + \sigma\alpha T + 2\tilde{b}\sigma + 2\tilde{b}\alpha$$

$$-rT - \cancel{\frac{1}{2}\alpha^2 T} - \cancel{\frac{2\tilde{b}^2}{T}}$$

$$= \frac{1}{2}\sigma^2 T + \cancel{\sigma}\overbrace{\frac{\left(r-\frac{1}{2}\sigma^2\right)}{\cancel{\sigma}}}^{\alpha}T + 2\frac{\log\left(\frac{b}{S_0}\right)}{\cancel{\sigma}}\cancel{\sigma} + 2\frac{\log\left(\frac{b}{S_0}\right)}{\sigma}\frac{\left(r-\frac{1}{2}\sigma^2\right)}{\sigma}$$

$$-rT$$

$$= \cancel{\frac{1}{2}\sigma^2 T} + \cancel{rT} - \cancel{\frac{1}{2}\sigma^2 T} + 2\log\left(\frac{b}{S_0}\right) + 2\left(r-\frac{1}{2}\sigma^2\right)\frac{1}{\sigma^2}\log\left(\frac{b}{S_0}\right) - \cancel{rT}$$

$$= 2\log\left(\frac{b}{S_0}\right) + 2\frac{r}{\sigma^2}\log\left(\frac{b}{S_0}\right) - \log\left(\frac{b}{S_0}\right)$$

$$= \log\left(\frac{b}{S_0}\right)\left(1+\frac{2r}{\sigma^2}\right) = \log\left(\frac{S_0}{b}\right)\left(-1-\frac{2r}{\sigma^2}\right)$$

$$\nu\sigma T = \left(\sigma + \alpha + \frac{2\delta}{T}\right)\sigma T$$

$$= \sigma^2 T + \frac{r - \frac{1}{2}\sigma^2}{\cancel{\sigma}}\cancel{\sigma} T + \frac{2}{\cancel{T}}\frac{\log\left(\frac{b}{S_0}\right)}{\cancel{\sigma}}\cancel{\sigma}\cancel{T}$$

$$= \underbrace{\sigma^2 T + rT - \frac{1}{2}\sigma^2 T}\ + 2\log\left(\frac{b}{S_0}\right)$$

$$= \left(r + \frac{1}{2}\sigma^2\right)T + 2\log\left(\frac{b}{S_0}\right)$$

$$\Downarrow$$

$$I_3 = \left(\frac{S_0}{b}\right)^{\left(-1-\frac{2r}{\sigma^2}\right)}\left(N\left(\frac{\log\left(\frac{S_0}{h}\right) + \left(r+\frac{1}{2}\sigma^2\right)T + 2\log\left(\frac{b}{S_0}\right)}{\sigma\sqrt{T}}\right) - N\left(\frac{\log\left(\frac{S_0}{b}\right) + \left(r+\frac{1}{2}\sigma^2\right)T + 2\log\left(\frac{b}{S_0}\right)}{\sigma\sqrt{T}}\right)\right)$$

$$= \left(\frac{S_0}{b}\right)^{-1-\frac{2r}{\sigma^2}}\left(N\left(\frac{\log\left(\frac{S_0 b^2}{h S_0^2}\right) + \left(r+\frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}}\right) - N\left(\frac{\log\left(\frac{S_0 b^2}{b S_0^2}\right) + \left(r+\frac{1}{2}r^2\right)T}{\sigma\sqrt{T}}\right)\right)$$

$$= \left(\frac{S_0}{b}\right)^{-1-\frac{2r}{\sigma^2}}\left(N\left(\delta_+\left(\frac{b^2}{S_0 h}\right)\right) - N\left(\delta_+\left(\frac{b}{S_0}\right)\right)\right)$$

$$\left(e^{\frac{1}{2}\gamma^2 T + \beta} \text{ and } \gamma\sigma T \right)$$

For I_4, $\beta = -rT - \frac{1}{2}\alpha^2 T - \frac{2\tilde{b}^2}{T}$, $\gamma = \alpha + \frac{2\tilde{b}}{T}$

$$\frac{1}{2}\gamma^2 T + \beta = \frac{1}{2}\overbrace{\left(\alpha^2 + 4\frac{\tilde{b}\alpha}{T} + 4\frac{\tilde{b}^2}{T^2}\right)}^{\gamma^2} T - rT - \frac{1}{2}\alpha^2 T - \frac{2\tilde{b}^2}{T}$$

$$= \cancel{\frac{1}{2}\alpha^2 T} + 2\tilde{b}\alpha + \cancel{2\frac{\tilde{b}^2}{T}} - rT - \cancel{\frac{1}{2}\alpha^2 T} - \cancel{\frac{2\tilde{b}^2}{T}}$$

$$= 2\,\frac{\log\left(\frac{b}{S_0}\right)}{\sigma}\,\frac{\left(r - \frac{1}{2}\sigma^2\right)}{\sigma} - rT$$

$$= \log\left(\frac{b}{S_0}\right)\left(\frac{2r}{\sigma^2} - 1\right) - rT$$

$$= \log\left(\frac{S_0}{b}\right)\left(1 - \frac{2r}{\sigma^2}\right) - rT$$

and

$$\gamma\sigma T = \left(\alpha + \frac{2\tilde{b}}{T}\right)\sigma T = \overbrace{\frac{r - \frac{1}{2}\sigma^2}{\sigma}}^{\alpha}\sigma T + \frac{2\log\left(\frac{b}{S_0}\right)}{\sigma T}\sigma T$$

$$= \left(r - \frac{1}{2}\sigma^2\right)T + 2\log\left(\frac{b}{S_0}\right)$$

$$I_4 = e^{-rT}\left(\frac{S_0}{b}\right)^{1-\frac{2r}{\sigma^2}}\left(N\left(\frac{\log\left(\frac{S_0}{k}\right) + \left(r-\frac{1}{2}\sigma^2\right)T + 2\log\left(\frac{b}{S_0}\right)}{\sigma\sqrt{T}}\right)\right.$$

$$\left. - N\left(\frac{\log\left(\frac{S_0}{b}\right) + \left(r-\frac{1}{2}\sigma^2\right)T + 2\log\left(\frac{b}{S_0}\right)}{\sigma\sqrt{T}}\right)\right)$$

$$= e^{-rT}\left(\frac{S_0}{b}\right)^{1-\frac{2r}{\sigma^2}}\left(N\left(\frac{\log\left(\frac{S_0 b^2}{S_0^2 k}\right) + \left(r-\frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}}\right)\right.$$

$$\left. - N\left(\frac{\log\left(\frac{S_0 b^2}{S_0^2 b}\right) + \left(r-\frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}}\right)\right)$$

$$= e^{-rT}\left(\frac{S_0}{b}\right)^{1-\frac{2r}{\sigma^2}}\left(N\left(\delta_-\left(\frac{b^2}{S_0 k}\right)\right) - N\left(\delta_-\left(\frac{b}{S_0}\right)\right)\right)$$

Hence,

$$V_0 = S_0 I_1 - k I_2 - S_0 I_3 + k I_4$$

$$= S_0\left(N\left(\delta_+\left(\frac{S_0}{k}\right)\right) - N\left(\delta_+\left(\frac{S_0}{b}\right)\right)\right)$$

$$- e^{-rT} k\left(N\left(\delta_-\left(\frac{S_0}{k}\right)\right) - N\left(\delta_-\left(\frac{S_0}{b}\right)\right)\right)$$

$$- b\left(\frac{S_0}{b}\right)^{-\frac{2r}{\sigma^2}}\left(N\left(\delta_+\left(\frac{b^2}{S_0 k}\right)\right) - N\left(\delta_+\left(\frac{b}{S_0}\right)\right)\right)$$

$$+ e^{-rT} k\left(\frac{S_0}{b}\right)^{1-\frac{2r}{\sigma^2}}\left(N\left(\delta_-\left(\frac{b^2}{S_0 k}\right)\right) - N\left(\delta_-\left(\frac{b}{S_0}\right)\right)\right)$$