Exercises: Extra material W6

1) Price model

$$dS_t = \mu S_t dt + \sigma S_t dW_t, \quad S_0 > 0$$

$\Downarrow$ - Ex. 2, W5

1.1)

$$S_T = S_0 \exp\left((\mu - \tfrac{1}{2}\sigma^2)T + \sigma W_T \right)$$

1.2) Compute the discounted expected payoffs of an European call and put.

Since $W_T \sim N(0,T)$

$$\sigma W_T + (\mu - \tfrac{1}{2}\sigma^2)T = Z + \mu T$$

$Z \sim N(-\frac{1}{2}\sigma^2 T, \sigma^2 T)$. Hence,

$$e^{-rT} E_P(S_T - K)_+ = e^{-rT} E_P(\underline{S_0 e^{Z+\mu T} - K})_+$$

$$S_0 e^{Z+\mu T} > K$$

$$\Updownarrow$$

$$Z > \log\left(\frac{K}{S_0}\right) - \mu T$$

Thus,

$$e^{-rT} E_P(S_T - K)_+ = \frac{1}{\sqrt{2\pi\sigma^2 T}} \int_{\log\left(\frac{K}{S_0}\right) - \mu T}^{\infty} \left(S_0 e^{z+(\mu-r)T} - K e^{-rT}\right) e^{-\frac{(z+\frac{1}{2}\sigma^2 T)^2}{2\sigma^2 T}} dz$$

Again

$$e^{-rT}E_P(S_T-K)_+ = \frac{1}{\sqrt{2\pi\sigma^2 T}}\int_{\log\left(\frac{K}{S_0}\right)-\mu T}^{\infty}\left(S_0 e^{z+(\mu-r)T}-Ke^{-rT}\right)e^{-\frac{(z+\frac{1}{2}\sigma^2 T)^2}{2\sigma^2 T}}\,dz$$

$$v = -\frac{z+\frac{1}{2}\sigma^2 T}{\sigma\sqrt{T}} \iff z = -\sigma\sqrt{T}\,v-\frac{1}{2}\sigma^2 T$$

$$\left|\frac{dz}{dv}\right| = \sigma\sqrt{T}$$

Hence, $e^{-rT}E_P(S_T-K)_+ =$

$$\frac{e^{(\mu-r)T}\cdot\sigma\sqrt{T}}{\sqrt{2\pi\sigma^2 T}}\int_{-\infty}^{\frac{\log\left(\frac{S_0}{K}\right)+(\mu-\frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}} S_0 e^{-\sigma\sqrt{T}v-\frac{1}{2}\sigma^2 T-\frac{1}{2}v^2}\,dv-\frac{Ke^{-rT}\sigma\sqrt{T}}{\sqrt{2\pi\sigma^2 T}}\int_{-\infty}^{\frac{\log\left(\frac{S_0}{K}\right)+(\mu-\frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}} e^{-\frac{1}{2}v^2}\,dv$$

Note $-\frac{1}{2}(v+\sigma\sqrt{T})^2 = -\frac{1}{2}v^2 - v\sigma\sqrt{T}-\frac{1}{2}\sigma^2 T$ and

use $w = v+\sigma\sqrt{T}$, $\left|\frac{dw}{dv}\right| = 1$ so that

$e^{-rT}E_P(S_T-K)_+ =$

$$\frac{S_0 e^{(\mu-r)T}}{\sqrt{2\pi}}\int_{-\infty}^{\frac{\log\left(\frac{S_0}{K}\right)+(\mu-\frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}+\sigma\sqrt{T}} e^{-\frac{1}{2}w^2}\,dw-\frac{Ke^{-rT}}{\sqrt{2\pi}}\int_{-\infty}^{\frac{\log\left(\frac{S_0}{K}\right)+(\mu-\frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}} e^{-\frac{1}{2}v^2}\,dv \quad \text{i.e.}$$

$e^{-rT}E_P(S_T-K)_+ =$

$$\boxed{S_0 e^{(\mu-r)T}N\left(\frac{\log\left(\frac{S_0}{K}\right)+(\mu+\frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}\right)-Ke^{-rT}N\left(\frac{\log\left(\frac{S_0}{K}\right)+(\mu-\frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}\right)}$$

For the put we note

$$e^{-rT}E_P(S_T-K) = e^{-rT}E_P((S_T-K)_+) - e^{-rT}E_P((K-S_T)_+)$$

$$\Downarrow$$

$$e^{-rT}E_P((K-S_T)_+) = \underbrace{e^{-rT}E_P((S_T-K)_+)}_{\text{see above}} + Ke^{-rT} - e^{-rT}E_P(S_T)$$

$$E_P(S_T) = \underbrace{S_0 e^{(\mu-\frac{1}{2}\sigma^2)T}\underbrace{E\left(e^{i(-i\sigma)W_T}\right)}_{= e^{-\frac{1}{2}(-i\sigma)^2T} = e^{\frac{1}{2}\sigma^2T}}}_{= S_0 e^{\mu T}}$$

Thus,

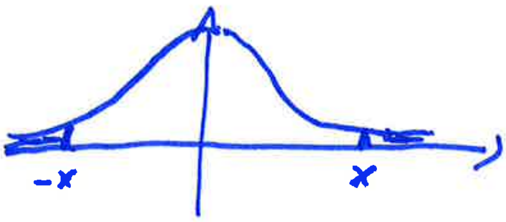
$$e^{-rT}E_P(K-S_T)_+ =$$

$$= S_0 e^{(\mu-r)T}N\left(\frac{\log\left(\frac{S_0}{K}\right)+(\mu+\frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}\right) - Ke^{-rT}N\left(\frac{\log\left(\frac{S_0}{K}\right)+(\mu-\frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}\right)$$

$$\underbrace{-S_0e^{(\mu-r)T} + Ke^{-rT}}$$

$$= -S_0e^{(\mu-r)T}\left(1-N\left(\frac{\log\left(\frac{S_0}{K}\right)+(\mu+\frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}\right)\right)$$

$$+Ke^{-rT}\left(1-N\left(\frac{\log\left(\frac{S_0}{K}\right)+(\mu-\frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}\right)\right)$$



Hence,

$e^{-rT}E_P((K-S_T)_+) =$

$$Ke^{-rT}N\left(-\frac{\log(\frac{S_0}{K})+(\mu-\frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}\right) - S_0e^{(\mu-r)T}N\left(-\frac{\log(\frac{S_0}{K})+(\mu+\frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}\right)$$

1.3) $C - P =$

$$\underbrace{\begin{aligned}&S_0e^{(\mu-r)T}N\left(\frac{\log(\frac{S_0}{K})+(\mu+\frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}\right) - Ke^{-rT}N\left(\frac{\log(\frac{S_0}{K})+(\mu-\frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}\right)\\ &+S_0e^{(\mu-r)T}\left(1-N\left(\frac{\log(\frac{S_0}{K})+(\mu+\frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}\right)\right) - Ke^{-rT}\left(1-N\left(\frac{\log(\frac{S_0}{K})+(\mu-\frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}\right)\right)\end{aligned}}$$

$$\parallel$$

$$S_0e^{(\mu-r)T} - Ke^{-rT}$$

But European PCP tells us

$$C - P = S_0 - Ke^{-rT} \neq S_0e^{(\mu-r)T} - Ke^{-rT}$$

except for $\mu = r$

5

2) Show for (M_t) - square integrable (non-neg.) martingale and $s \leq t$

$$E((M_t - M_s)^2 | \mathcal{F}_s) = E(M_t^2 - M_s^2 | \mathcal{F}_s)$$

Note that $E(M_t M_s) \leq \sqrt{E(M_t^2) E(M_s^2)} < \infty$

Furthermore,

$$\underline{E((M_t - M_s)^2 | \mathcal{F}_s)} = E(M_t^2 - 2M_t M_s + M_s^2 | \mathcal{F}_s)$$

$$= E(M_t^2 | \mathcal{F}_s) - \underbrace{2M_s \underbrace{E(M_t | \mathcal{F}_s)}_{M_s} + M_s^2}_{-M_s^2}$$

$$= \underline{E(M_t^2 - M_s^2 | \mathcal{F}_s)}$$

3) Recall ... $\Rightarrow$

$$df(t,X_t) = \left(a_t \frac{\partial f}{\partial x}(t,X_t) + \frac{\partial f}{\partial t}(t,X_t) + \frac{1}{2} b_t^2 \frac{\partial^2 f}{\partial x^2}(t,X_t)\right)dt$$

$$b_t \frac{\partial f}{\partial x}(t,X_t)\,dW_t, \quad dX_t = a_t\,dt + b_t\,dW_t$$

$$d(\sin(W_t)), \quad dX_t = dW_t = \underbrace{0}_{a_t}dt + \underbrace{1}_{b_t}dW_t$$

$$f(x) = \sin(x) \Rightarrow \frac{\partial f}{\partial x}(x) = \cos(x), \quad \frac{\partial^2 f}{\partial x^2}(x) = -\sin(x)$$

$$\frac{\partial f}{\partial t}(x) = 0$$

Hence,

$$d(\sin(W_t)) = \underbrace{-\frac{1}{2}\sin(W_t)}_{\left(0 + 0 + \frac{1}{2}\cdot 1^2(-\sin(W_t))\right)}dt + \cos(W_t)\,dW_t$$

$$d(e^{W_t^2}), \quad dX_t = dW_t = \underbrace{0}_{a_t}dt + \underbrace{1}_{b_t}\cdot dW_t$$

$$F(x) = e^{x^2}, \quad \frac{\partial F}{\partial x}(x) = 2x e^{x^2}, \quad \frac{\partial^2 F}{\partial x^2}(x) = 2e^{x^2} + 4x^2 e^{x^2}$$

$$\frac{\partial F}{\partial t}(x) = 0$$

Thus,

$$d(e^{W_t^2}) = \frac{1}{2}\cdot 1^2\left(2e^{W_t^2} + 4W_t^2 e^{W_t^2}\right)dt + 2W_t e^{W_t^2}\,dW_t$$

$$= e^{W_t^2}\left((1+2W_t^2)\,dt + 2W_t\,dW_t\right)$$

⑦

$d(e^{tW_t})$, $\quad dX_t = dW_t = \underbrace{0}_{a_t}dt + \underbrace{1}_{b_t}dW_t$

$$F(t,x) = e^{tx}, \quad \frac{\partial F}{\partial x}(t,x) = te^{tx}, \quad \frac{\partial^2 F}{\partial x^2}(t,x) = t^2 e^{tx}$$

$$\frac{\partial F}{\partial t}(t,x) = xe^{tx}$$

$$\Downarrow$$

$$\underline{d(e^{tW_t})} = \left(0 + W_t e^{tW_t} + \frac{1}{2}1^2 t^2 e^{tW_t}\right)dt + 1 te^{tW_t}dW_t$$

$$= \underline{e^{tW_t}\left(\left(W_t + \frac{1}{2}t^2\right)dt + t\,dW_t\right)}$$

Furthermore, consider

$$(X_t) = \left(\frac{\exp(\sigma W_t)}{1+t}\right)$$

$$Y_t = \sigma W_t, \quad dY_t = \underbrace{0}_{a_t}dt + \underbrace{\sigma}_{b_t}dW_t$$

$$X_t = F(t, Y_t) = \frac{\exp(Y_t)}{1+t}, \quad F(t,y) = \frac{e^y}{1+t}$$

$$\frac{\partial F}{\partial y}(t,y) = \underbrace{\frac{\exp(y)}{1+t}}_{\substack{\parallel \\ F(t,y)}} = \frac{\partial^2 F}{\partial y^2}(t,y)$$

$F(t,y)$, where $F(t,Y_t) = X_t$

$$\frac{\partial F}{\partial t}(t,y) = -\frac{\exp(y)}{(1+t)^2} \implies \frac{\partial F}{\partial t}(t,Y_t) = -\underbrace{\frac{\exp(Y_t)}{1+t}}_{X_t}\cdot\frac{1}{1+t}$$

Hence,

$$\underset{\sim}{dX_t} = \left(0 \cdot X_t - \frac{X_t}{1+t} + \frac{1}{2}\sigma^2 X_t\right)dt + \sigma X_t \, dW_t$$

$$= \underline{X_t\left(\left(\frac{1}{2}\sigma^2 - \frac{1}{1+t}\right)dt + \sigma \, dW_t\right)},$$

thus SDE is satisfied.

4) European power-call price

$$e^{-rT} E_Q\left(\left(S_T - K\right)_+^2\right)$$

important

Message: We can find quite easily prices of, at first glance, quite complicated contracts.

We know $S_T = S_0 e^{(r - \frac{1}{2}\sigma^2)T + \sigma W_T}$

where (W_t) - is a Q - sBM.

$$E_Q\left((S_T-k)_+^n\right) = E_Q\left((S_0 e^{Z+rT}-k)_+^n\right)$$ where

$$\sigma W_T + (r-\tfrac{1}{2}\sigma^2)T = Z + rT$$

$$W_T \sim N(0,T) \text{ i.e. } Z \sim N(-\tfrac{1}{2}\sigma^2 T, \sigma^2 T)$$

Note that $S_0 e^{Z+rT} - k \geq 0$

$$\Updownarrow$$

$$Z \geq \log\left(\frac{k}{S_0}\right) - rT$$

$$=$$

$$\frac{1}{\sqrt{2\pi\sigma^2 T}} \int_{\log\left(\frac{k}{S_0}\right)-rT}^{\infty} (S_0 e^{z+rT}-k)^n \exp\left(-\frac{1}{2}\frac{(z+\frac{1}{2}\sigma^2 T)^2}{\sigma^2 T}\right) dz$$

$$\parallel$$

$$\frac{1}{\sqrt{2\pi\sigma^2 T}} \int_{\log\left(\frac{k}{S_0}\right)-rT}^{\infty} \sum_{j=0}^{n} (-1)^j \binom{n}{j} S_0^{n-j} e^{(n-j)(z+rT)} k^j \exp\left(-\frac{1}{2}\frac{(z+\frac{1}{2}\sigma^2 T)^2}{\sigma^2 T}\right) dz$$

$$\parallel$$

$$\frac{1}{\sqrt{2\pi\sigma^2 T}} \sum_{j=0}^{n} (-1)^j \binom{n}{j} S_0^{n-j} e^{(n-j)rT} k^j \int_{\log\left(\frac{k}{S_0}\right)-rT}^{\infty} \exp((n-j)z) \exp\left(-\frac{1}{2}\frac{(z+\frac{1}{2}\sigma^2)^2}{\sigma^2 T}\right) dz$$

$$v = -\frac{z+\frac{1}{2}\sigma^2 T}{\sigma\sqrt{T}} \iff z = -\sigma\sqrt{T}\, v - \frac{1}{2}\sigma^2 T$$

$$\left|\frac{dz}{dv}\right| = \sigma\sqrt{T}$$

Thus, $E_Q((S_T-k)_+^n)$

$$\Vert$$

$$\frac{\sigma\sqrt{T}}{\sqrt{2\pi\sigma^2 T}}\sum_{j=0}^{n}(-1)^j\binom{n}{j}S_0^{n-j}e^{(n-j)rT}k^j\int_{-\infty}^{\frac{\log\left(\frac{S_0}{k}\right)+rT-\frac{1}{2}\sigma^2 T}{\sigma\sqrt{T}} = d_-}\exp\left(+\underbrace{(n-j)\left(-\sigma\sqrt{T}v-\frac{1}{2}\sigma^2 T\right)-\frac{1}{2}v^2}_{-(n-j)\sigma\sqrt{T}v-(n-j)\frac{1}{2}\sigma^2 T-\frac{1}{2}v^2}\right)dv$$

Note $-\frac{1}{2}\left(v+(n-j)\sigma\sqrt{T}\right)^2=-\frac{1}{2}\left(v^2+2(n-j)\sigma\sqrt{T}v+(n-j)^2\sigma^2 T\right)$

$$\Downarrow$$

$$\text{exponent}=-\frac{1}{2}\left(v+(n-j)\sigma\sqrt{T}\right)^2\underbrace{\underbrace{+\frac{1}{2}(n-j)^2\sigma^2 T}_{\text{compensation}}\underbrace{-\frac{1}{2}(n-j)\sigma^2 T}_{\text{what is in the exponent}}}_{\frac{1}{2}\sigma^2 T(n-j)(n-j-1)}$$

Hence, $\underline{E_Q((S_T-k)_+^n)}$

$$\Vert$$

$$\sum_{j=0}^{n}(-1)^j\binom{n}{j}S_0^{n-j}e^{(n-j)rT}e^{\frac{\sigma^2 T}{2}(n-j)(n-j-1)}k^j\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{d_-}e^{-\frac{1}{2}\left(v+(n-j)\sigma\sqrt{T}\right)^2}dv$$

$w=v+(n-j)\sigma\sqrt{T}$, $v=w-(n-j)\sigma\sqrt{T}$, thus,

$E_Q((S_T-k)_+^n)=$ $\qquad\left|\frac{dw}{dv}\right|=1$

$$=\underline{\sum_{j=0}^{n}(-1)^j\binom{n}{j}S_0^{n-j}e^{(n-j)\left(rT+\frac{\sigma^2 T}{2}(n-j-1)\right)}k^j N\left(d_-+(n-j)\sigma\sqrt{T}\right)}$$

By discounting we obtain the price.

5) Some background: Roughly:

For a stochastic process we write $(\xi(t,\omega))$. For each fixed ω the function $\xi(t,\omega)$, $t \in T$ - time domain, is called a realisation of the stochastic process. The values of ξ are assumed to belong to a metric space $X = S$. Then ξ is a random element in the space $X = S^T$ of S-valued functions with arguments from T.

From now on $S = \mathbb{R}$ or $\mathbb{R}^d$.

In order to generate a σ-algebra on S^T we may use evaluation maps

$$\pi_t : S^T \longrightarrow S$$

$$x \longrightarrow \pi_t(x) = x(t), \quad t \in T$$

$\mathcal{L} = \sigma\{\pi_t, t \in T\}$ - cylinderial σ-algebra is the minimal σ-algebra which makes all evaluation maps π_t measurable, $\xi_t = \pi_t \circ \xi$.

Some technical problems can be settled by requiring that all values ξ_t can be explored by using only a countable dense set of time moments (seperable processes).

Every stochastic process has an equivalent seperable version in the sense

η is a version of ξ if

$$P(\xi(t)=\eta(t)) = 1 \quad \forall t \in T$$

(need not have the same sample paths).

We consider these versions.

(Recall that a version is indistinguishable if $\quad P(\xi(t)=\eta(t), \forall t \in T) = 1$)

The finite dimensional distribution $\mu_{t_1,\dots,t_n}$ is the distribution of the random element $(\xi(t_1),\dots,\xi(t_n))$ which takes values in the space $S^n = S\times\dots\times S$ with $\mathcal{B}(S^n)$.

Prop: Let ξ and η be two stochastic processes. Then $\xi \overset{d}{=} \eta$

$$\Updownarrow$$

$(\xi_{t_1}, \dots, \xi_{t_n})$ and $(\eta_{t_1}, \dots, \eta_{t_n})$ coincide in distribution for all $t_1, \dots, t_n \in T$ and all $n \geq 1$.

Since ξ in Ex. 5 is real-valued, the finite dimensional distribution $\mu_{t_1, \dots, t_n}$ is defined by means of the corresponding multivariable cumulative distribution function

$$F_{t_1, \dots, t_n}(x_1, \dots, x_n) = P(\xi_{t_1} \leq x_1, \dots, \xi_{t_n} \leq x_n)$$

$$\forall\, x_1, \dots, x_n \in \mathbb{R},\ n \geq 1$$

Assume that for ξ the $(\xi_{t_1}, \dots, \xi_{t_n})$ have a Gaussian distribution with mean vector $(m(t_1), \dots, m(t_n))$ and covariance matrix $(K(t_j, t_k))_{j,k=1}^{n}$. Then ξ is a Gaussian process.

(Notice that in this case we know the distribution of the process if we know $E(\xi_t)\ \forall t$ and $\mathrm{Cov}(\xi_t,\xi_s)\ \forall t,s$)

Further result from probability th. / math. stat.:

$(\xi_{t_1},\dots,\xi_{t_n})$ jointly gaussian

$\Updownarrow$

for all linear combinations $\lambda_1\xi_{t_1}+\dots+\lambda_n\xi_{t_n}$ is a normally distributed random variable.

5.1) Ornstein-Uhlenbeck

$$dX_t = \underbrace{-cX_t}_{a(t,X_t)=-cX_t}\,dt + \underbrace{\sigma}_{b(t,X_t)=\sigma}\,dW_t,\quad X_0 = x \qquad (*)$$

We have to check that there is a $K<\infty$, such that

$|a(t,x)-a(t,y)| + |b(t,x)-b(t,y)| \le K|x-y|$

$|a(t,x)| + |b(t,x)| \le K(1+|x|) \qquad \forall t,x,y$

(given that the starting point is deterministic)

We have

$$
\begin{array}{ccccc}
|a(t,x)-a(t,y)| & + & |b(t,x)-b(t,y)| & & \\
\parallel & & \parallel & & \\
|-cx-(-cy)| & + & |\sigma-\sigma| & = & |c||x-y|
\end{array}
$$

and

$$
\begin{array}{ccccc}
|a(t,x)| & + & |b(t,x)| & & \\
\parallel & & \parallel & & \\
|-cx| & + & |\sigma| & \leq & \max(|c|,|\sigma|)(1+|x|)
\end{array}
$$

$$K = \max(|c|,|\sigma|) < \infty$$

$$\Downarrow$$

Hence, (*) admits a unique (strong) solution.

5.2) We show that this solution is given by

$$X_t = e^{-ct}x + \sigma e^{-ct}\underbrace{\int_0^t e^{cs}\,dW_s}_{Y_t = \int_0^t e^{cs}\,dW_s}$$

$$\boxed{\begin{aligned}&\text{Recall} \ldots\\ &df(t,X_t) = \left(a_t \frac{\partial f}{\partial x}(t,X_t) + \frac{\partial f}{\partial t}(t,X_t) + \frac{1}{2}b_t^2\frac{\partial^2 f}{\partial x^2}(t,X_t)\right)dt\\ &\qquad + \frac{\partial f}{\partial x}(t,X_t)\,b_t\,dW_t, \quad dX_t = a_t\,dt + b_t\,dW_t\end{aligned}}$$

$$dY_t = \underbrace{0}_{a_t}\cdot dt + \underbrace{e^{ct}}_{b_t}\,dW_t$$

$$X_t = f(t,Y_t) = e^{-ct}x + \sigma e^{-ct}Y_t$$

$$f(t,y) = e^{-ct}x + \sigma e^{-ct}y,$$

$$\frac{\partial f}{\partial y}(t,y) = \sigma e^{-ct}, \quad \frac{\partial^2 f}{\partial y^2}(t,y) = 0,$$

$$\frac{\partial f}{\partial t}(t,y) = -cxe^{-ct} - \sigma c y e^{-ct}, \quad \text{thus,}$$

$$dX_t = (0 - cxe^{-ct} - \sigma c Y_t e^{-ct} + 0)\,dt + \sigma e^{-ct}\underbrace{e^{ct}}_{b_t}\,dW_t$$

$$= -c\underbrace{\Big(xe^{-ct} + \sigma e^{-ct}\underbrace{\int_0^t e^{cs}\,dW_s}_{Y_t}\Big)}_{X_t}dt + \sigma\,dW_t$$

Hence,

$$dX_t = -cX_t\,dt + \sigma\,dW_t \,, \text{ where } X_0 = x$$

5.3) In view of the provided background consider

$$X_{t_i} = x e^{-ct_i} + \sigma e^{-ct_i} \int_0^T 1\!\!1_{\{s \le t_i\}} e^{cs}\,dW_s$$

Hence,

$$X_{t_i} = m_i + \int_0^T f_i(s)\,dW_s$$

where $0 \le f_i$ - dominated by a continuous function, which is bounded on the compact intervall $[0,T]$, and deterministic

Consider f being deterministic and bounded

$$\Downarrow \quad (\text{e.g. Novikov})$$

$$\left(\exp\left(\int_0^t u f(s)\,dW_s - \frac{1}{2}\int_0^t (u f(s))^2\,ds\right)\right)_{t \in [0,T]}$$

$u \in \mathbb{R}$

is a (true) martingale with expectation 1 at $t=0$.

With $I(t) = \int_0^t f(s)\,dW_s$ (18)

$$E\left(\exp\left(u\, I(t) - \frac{1}{2}\int_0^t (u f(s))^2 ds\right)\right) = 1$$

$$\Downarrow$$

$$E(\exp(u\, I(t))) = \exp\left(+\frac{1}{2}u^2 \int_0^t f(s)^2 ds\right) \quad \forall u \in \mathbb{R}$$

$$\Downarrow$$

$$I(t) \sim N\left(0, \int_0^t f(s)^2 ds\right).$$

Thus, we also have that

$$\lambda_1 X_{t_1} + \dots + \lambda_n X_{t_n} = \sum_{i=1}^n \lambda_i m_i + \int_0^T \left(\sum_{i=1}^n \lambda_i f_i(s)\right) dW_s$$

is a gaussian random variable for all $\lambda \in \mathbb{R}^n$

$$\Downarrow$$

$(X_{t_1}, \dots, X_{t_n})$ is a random vector with (joint) gaussian distribution $\forall\, t_1, \dots, t_n, \forall n$

$$\Downarrow$$

(X_t) is a gaussian process.

Hence, if we know $E(X_t)\ \forall t$ and $Cov(X_t, X_s)\ \forall t, s$ we have characterised its law.

Note that $E\left(\int_0^T (e^{cs})^2 ds\right) < \infty$

$$\Downarrow$$

$$\left(\int_0^t e^{cs} dW_s\right) = (M_t) \text{ is a}$$

(square integrable) martingale with $M_0 = 0$

$$\Downarrow$$

$$E(X_t) = E\left(xe^{-ct} + \sigma e^{-ct}\int_0^t e^{cs} dW_s\right)$$

$$= xe^{-ct} + \sigma e^{-ct} \underbrace{E\left(\int_0^t e^{cs} dW_s\right)}_{=0}$$

$$= xe^{-ct}.$$

Since

$$E\left(e^{u\left(\sigma e^{-ct}\int_0^t e^{cs}\,dW_s\right)}\right) = e^{\frac{1}{2}u^2\sigma^2 e^{-2ct}\int_0^t e^{2cs}\,ds}$$

$$\Downarrow$$

$$Var(X_t) = \sigma^2 e^{-2ct}\int_0^t e^{2cs}\,ds$$

$$= \sigma^2 e^{-2ct}\left(\frac{e^{2cs}}{2c}\Big|_0^t\right)$$

$$= \frac{\sigma^2 e^{-2ct}}{2c}\left(e^{2ct}-1\right)$$

$$= \frac{\sigma^2}{2c}\left(1-e^{-2ct}\right)$$

21

$$\mathrm{Cov}(X_s,X_t) = E\Big(\underbrace{(X_s - E(X_s))}_{xe^{-cs}+\sigma e^{-cs}\int_0^s e^{cu}\,dW_u - xe^{-cs}}\underbrace{(X_t - E(X_t))}_{xe^{-ct}+\sigma e^{-ct}\int_0^t e^{cu}\,dW_u - xe^{-ct}}\Big)$$

$$= E\Big(\Big(\sigma e^{-cs}\int_0^s e^{cu}\,dW_u\Big)\Big(\sigma e^{-ct}\int_0^t e^{cu}\,dW_u\Big)\Big)$$

$$= \sigma^2 e^{-c(s+t)}\, \underbrace{E\Big(\int_0^s e^{cu}\,dW_u \int_0^t e^{cu}\,dW_u\Big)}$$

$\shortparallel$ for $s \le t$

$$E\Big(\underbrace{\int_0^s e^{cu}\,dW_u \int_0^s e^{cu}\,dW_u}_{=} + \underbrace{\int_0^s e^{cu}\,dW_u \int_s^t e^{cu}\,dW_u}_{E(\;\;)=0}\Big)$$

independent increments

Itô isometry

$$\underbrace{E\Big(\int_0^s e^{2cu}\,du\Big)}_{=}$$

$$\underbrace{\frac{e^{2cu}}{2c}\Big|_0^s = \frac{e^{2cs}-1}{2c}}$$

analogously for $s > t$ $\quad \dfrac{e^{2ct}-1}{2c}$

$$= \frac{\sigma^2 e^{-c(s+t)}}{2c}\left(e^{2c\min(s,t)} - 1\right)$$

5.4) The $(\mathcal{F}_t)$-adapted process (X_t) satisfies the Markov property if for any bounded Borel function f and for any s and t such that $s \leq t$, we have

$$E(f(X_t) | \mathcal{F}_s) = E(f(X_t) | \sigma(X_s))$$

Result: SDEs with "sufficiently regular" parameters are Markov processes.

a bit heuristically

Since $dX_t = -cX_t\,dt + \sigma\,dW_t$

does not (directly) depend on t

$$\Downarrow$$

$$X_{t+s} | X_s = x \sim X_t | X_0 = x$$

For the simulation method, see the soutions.

5.5) $X_0 \sim N(0, \sigma_0^2)$ $\quad X_0 \perp\!\!\!\perp (W_t)$

Find σ_0 such that the law of X_t does not depend on t.

$$X_t = \underbrace{X_0 e^{-ct}}_{X_0\text{- gaussian}} + \underbrace{\sigma e^{-ct}\int_0^t e^{cs}\,dW_s}_{\text{gaussian}}$$

independent

$\Downarrow$

(joint distribution of them is gaussian $\Rightarrow$ sum of them is gaussian)

$$Var(X_t) = e^{-2ct}\,Var(X_0) + \frac{\sigma^2}{2c}\left(1 - e^{-2ct}\right)$$

$$E(X_t) = 0 \qquad (0+0)$$

We want

$$e^{-2ct}\,Var(X_0) + \frac{\sigma^2}{2c}\left(1 - e^{-2ct}\right) = K > 0 \;\; \forall t$$

Thus,

$$e^{-2ct}\,Var(X_0) = K - \frac{\sigma^2}{2c}\left(1 - e^{-2ct}\right)$$

$$\Updownarrow$$

$$Var(X_0) = \underbrace{e^{2ct}K - \frac{\sigma^2}{2c}e^{2ct}}_{=\, e^{2ct}\left(\underbrace{K - \frac{\sigma^2}{2c}}_{=\,0}\right)} + \frac{\sigma^2}{2c}$$

– such that the variance does not depend on t (otherwise it would depend on t)

$$\Downarrow$$

$$K = \frac{\sigma^2}{2c}$$

Hence, $Var(X_0) = \dfrac{\sigma^2}{2c}$

$$\underline{X_0 \sim N\left(0, \frac{\sigma^2}{2c}\right)}$$