Exercises: Extra material WS, Ex. 2

2) Consider

$$S_t = x_0 + \int_0^t \mu S_u du + \int_0^t \sigma S_u dW_u$$

often written symbolically as

$$dS_t = \mu S_t dt + \sigma S_t dW_t, \quad x_0 > 0 \quad (*)$$

2.1) Show that

$$S_t = x_0 e^{(\mu - \frac{\sigma^2}{2})t + \sigma W_t}, \quad t \in [0,T]$$

is a solution of $(*)$.

Consider $X_t = (\mu - \frac{\sigma^2}{2})t + \sigma W_t$

$$\Rightarrow dX_t = \underbrace{(\mu - \tfrac{1}{2}\sigma^2)}_{a_t} dt + \underbrace{\sigma}_{b_t} dW_t$$

and $S_t = x_0 e^{X_t}$, $S(t,x) = S(x) = x_0 e^x$

$$\Rightarrow \frac{\partial S}{\partial x}(x) = S(x) = \frac{\partial^2 S}{\partial x^2}(x);$$

$$\frac{\partial S}{\partial t}(x) = 0$$

Hence, $S(t, X_t) = \frac{\partial S}{\partial x}(X_t) = \frac{\partial^2 S}{\partial x^2}(X_t) = S_t$

Recall $df(t,X_t) = \left(a_t \frac{\partial f}{\partial x}(t,X_t) + \frac{\partial f}{\partial t}(t,X_t) + \frac{1}{2} b_t^2 \frac{\partial^2 f}{\partial x^2}(t,X_t)\right)dt + b_t \frac{\partial f}{\partial x}(t,X_t)\, dW_t$

$\Downarrow$ Itô

$$dS_t = \Big(\underbrace{(\mu - \tfrac{\sigma^2}{2})}_{a_t} \underbrace{S_t}_{\frac{\partial S}{\partial x}(X_t)} + 0 + \tfrac{1}{2}\underbrace{\sigma^2}_{b_t^2} \underbrace{S_t}_{\frac{\partial^2 S}{\partial x^2}(X_t)}\Big)dt + \underbrace{\sigma}_{b_t} \underbrace{S_t}_{\frac{\partial S}{\partial x}(X_t)} dW_t$$

$$= \mu S_t\, dt + \sigma S_t\, dW_t$$

2.2) $S_0 = x_0$, consider

$$Z_t = \frac{S_0}{S_t} = \exp\left(\left(-\mu + \frac{\sigma^2}{2}\right)t - \sigma W_t\right), \quad t \in [0,T]$$

$$X_t = \left(-\mu + \frac{\sigma^2}{2}\right)t + (-\sigma) W_t$$

$$dX_t = \underbrace{\left(-\mu + \frac{\sigma^2}{2}\right)}_{a_t} dt + \underbrace{(-\sigma)}_{b_t} dW_t$$

$$Z_t = f(X_t) = e^{X_t}, \quad f(x) = e^x \Rightarrow \frac{\partial f}{\partial x}(x) = f(x) = \frac{\partial^2 f}{\partial x^2}(x)$$

$$\frac{\partial f}{\partial t}(x) = 0, \quad \frac{\partial f}{\partial x}(X_t) = f(X_t) = \frac{\partial^2 f}{\partial x^2}(X_t) = Z_t$$

$\swarrow$ Itô

$$dZ_t = \left(\left(-\mu + \frac{\sigma^2}{2}\right)Z_t + 0 + \frac{1}{2}\sigma^2 Z_t\right)dt + (-\sigma) Z_t\, dW_t, \text{ i.e.}$$

$$Z_t = \underbrace{1}_{= Z_0} + \int_0^t \underbrace{(\sigma^2 - \mu)}_{\mu'} Z_u\, du + \int_0^t \underbrace{(-\sigma)}_{\sigma'} Z_u\, dW_u$$

2.3) Show: If (Y_t) also solves the SDE then it coincides with (S_t)

Hence,

$$dY_t = \overbrace{\mu Y_t}^{a_t}\,dt + \overbrace{\sigma Y_t}^{b_t}\,dW_t$$

$$dZ_t = \underbrace{(\sigma^2-\mu)Z_t}_{a_t'}\,dt + \underbrace{(-\sigma)Z_t}_{b_t'}\,dW_t$$

$$\langle Y,Z\rangle_t = \int_0^t b_s b_s'\,ds = \int_0^t \sigma Y_s(-\sigma)Z_s\,ds\,;$$

$$d\langle Y,Z\rangle_t = -\sigma^2 Y_t Z_t\,dt$$

Consider

$$d(Y_tZ_t) = Y_t\,dZ_t + Z_t\,dY_t + d\langle Y,Z\rangle_t$$

$$= Y_t\,dZ_t + Z_t\,dY_t - \sigma^2 Z_t Y_t\,dt$$

$$= Y_t(\sigma^2-\mu)Z_t\,dt + Y_t(-\sigma)Z_t\,dW_t + Z_t\mu Y_t\,dt + Z_t\sigma Y_t\,dW_t - \sigma^2 Z_t Y_t\,dt$$

$$= Y_tZ_t(\underbrace{\sigma^2-\mu+\mu-\sigma^2}_{=0})\,dt + Y_tZ_t(\underbrace{-\sigma+\sigma}_{=0})\,dW_t$$

$$= 0$$

$$\Rightarrow Y_tZ_t = \underset{\substack{\| \\ x_0 = S_0}}{Y_0}\,\underset{=1}{Z_0}$$

$$\Rightarrow Y_t = S_0 Z_t^{-1} = S_0\left(\frac{S_t}{S_0}\right) = S_t \quad \forall t\in[0,T] \ P\text{-a.s.}$$

Alternatively, theory from SDE

$$dS_t = \underbrace{\mu S_t dt}_{a(t,S_t) = a(S_t) = \mu S_t} + \underbrace{\sigma S_t dW_t}_{b(t,S_t) = b(S_t) = \sigma S_t}$$

If with deterministic starting point

- $|a(t,x) - a(t,y)| + |b(t,x) - b(t,y)| \leq K|x-y|$ and
- $|a(t,x)| + |b(t,x)| \leq K(1+|x|)$ $\quad \forall\, t, x, y$ for a $K < \infty$

then the SDE has a unique (strong) solution

Note:

- $|\mu x - \mu y| + |\sigma x - \sigma y| = |\mu||x-y| + |\sigma||x-y|$
  $\leq 2\max(|\mu|, |\sigma|)|x-y|$ and
- $|\mu x| + |\sigma x| = |\mu||x| + |\sigma||x|$
  $\leq 2\max(|\mu|, |\sigma|) \cdot |x| \leq 2\max(|\mu|, |\sigma|)(1+|x|)$
  $\forall\, x, y \ (t)$

$\Downarrow$

SDE has a unique (strong) solution (S_0 given)