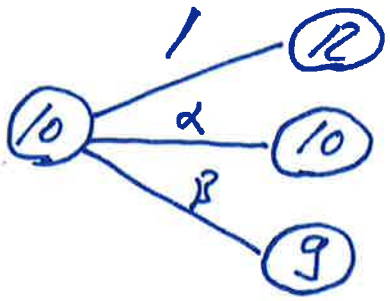
Exercises: Extra material W4

1) a non-complete example



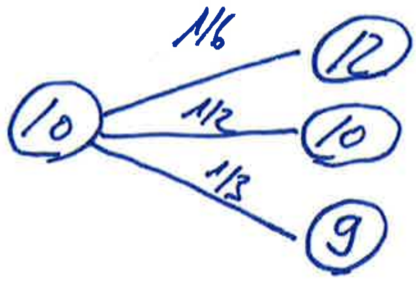
$r = 0$

$H(S_1)$ defines a financial derivative.

1.1) Is this market viable?

Consider Q being defined by

$\{\beta, \alpha, \gamma\} = \{1/3, 1/2, 1/6\}$ i.e.



$$E(S_1) = \frac{1}{3}\cdot 9 + \frac{1}{2}\cdot 10 + \frac{1}{6}\cdot 12 = 3+5+2 = 10 = S_0$$

$\Downarrow$

Hence, at least 1 martingale measure exists on this tree.

1.2) Characterise the risk-neutral probabilities.

Let Q be defined by (α, β, γ).
We need that

- $0 < \alpha, \beta, \gamma < 1$ ⋯ in order to get an "equivalent" probability measure
- $\gamma + \beta + \alpha = 1$
- $12 \cdot \gamma + 9 \cdot \beta + 10 \cdot \alpha = 10$ ⋯ martingale property

solve

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 12 & 9 & 10 & 10 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & -3 & -2 & -2 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 1 & 2/3 & 2/3 \end{array}\right)$$

$$\sim \left(\begin{array}{ccc|c} 1 & 0 & 1/3 & 1/3 \\ 0 & 1 & 2/3 & 2/3 \end{array}\right) \quad \text{i.e.} \quad \begin{aligned} 1\gamma + 1/3\,\alpha &= 1/3 \\ 1 \cdot \beta + 2/3\,\alpha &= 2/3 \end{aligned}$$

Hence,

$$\left.\begin{aligned} \gamma &= 1/3 - 1/3\,\alpha \\ \beta &= 2/3 - 2/3\,\alpha \\ \alpha &= \alpha \quad \text{for } \alpha \in (0,1) \end{aligned}\right\} \quad \text{all } \in (0,1) \text{ for } \alpha \in (0,1)$$

1.3) Replication in this market

We want

$$\phi^0 S_1^0 + \phi S_1 = H(S_1)$$

(ϕ^0, ϕ) describes the replicating strategy for this (1) period.

Hence,

$$\phi^0 + 9\phi = H(9)$$
$$\phi^0 + 10\phi = H(10)$$
$$\phi^0 + 12 \cdot \phi = H(12)$$

$$\begin{pmatrix} 1 & 9 \\ 1 & 10 \\ 1 & 12 \end{pmatrix} \begin{pmatrix} \phi^0 \\ \phi \end{pmatrix} = \begin{pmatrix} H(9) \\ H(10) \\ H(12) \end{pmatrix}$$

i.e. more equations than degrees of freedom.

A solution exists iff

$$\operatorname{rank}\begin{pmatrix} 1 & 9 \\ 1 & 10 \\ 1 & 12 \end{pmatrix} = \operatorname{rank}\begin{pmatrix} 1 & 9 & H(9) \\ 1 & 10 & H(10) \\ 1 & 12 & H(12) \end{pmatrix}$$

Note

$$\begin{pmatrix} 1 & 9 & H(9) \\ 1 & 10 & H(10) \\ 1 & 12 & H(12) \end{pmatrix} \sim \begin{pmatrix} 1 & 9 & H(9) \\ 1 & 10 & H(10) \\ 0 & 3 & H(12)-H(9) \end{pmatrix} \sim \begin{pmatrix} 1 & 9 & H(9) \\ 0 & 1 & H(10)-H(9) \\ 0 & 3 & H(12)-H(9) \end{pmatrix}$$

$$\sim \begin{pmatrix} 1 & 9 & H(9) \\ 0 & 1 & H(10)-H(9) \\ 0 & 0 & \underbrace{H(12)-H(9)-3H(10)+3H(9)}_{= 0} \end{pmatrix}$$

$$\operatorname{rank}\begin{pmatrix} 1 & 9 \\ 1 & 10 \\ 1 & 12 \end{pmatrix} = 2$$

This is needed, s.t. $\operatorname{rank}\begin{pmatrix} 1 & 9 & H(9) \\ 1 & 10 & H(10) \\ 1 & 12 & H(12) \end{pmatrix} = 2$

Hence, H can be replicated, iff

$$H(12) - 3H(6) + 2H(9) = 0$$

1.4) We show that for replicable options the value does not depend on the choice of the martingale measure.

Consider two possible α, α'; $\alpha \neq \alpha'$.
We show that $E_{Q^{\alpha}}(H) = E_{Q^{\alpha'}}(H)$

$$E_{Q^{\alpha}}(H) = \underbrace{\frac{2}{3}(1-\alpha)}_{\beta} \cdot H(9) + \alpha H(6) + \underbrace{\frac{1}{3}(1-\alpha)}_{\gamma} H(12)$$

$$= \frac{2}{3}(1-\alpha)H(9) + \alpha H(6) + \frac{1}{3}(1-\alpha)H(12) - 0$$

$$\frac{\alpha'-\alpha}{3}\underbrace{(H(12)-3H(6)+2H(9))}_{= 0 \text{ since } H\text{-replicable}}$$

$$= \frac{2}{3}(1-\alpha')H(9) + \alpha' H(6) + \frac{1}{3}(1-\alpha')H(12) = E_{Q^{\alpha'}}(H)$$

1.5) Describe the product satisfying $H(12) - 3H(10) + 2H(9) = 0$, i.e the replicable product.

Consider,

$$\frac{H(10)-H(9)}{10-9}, \quad \frac{H(12)-H(10)}{12-10}, \quad \frac{H(12)-H(9)}{12-9}$$

where $H(12) = 3H(10) - 2H(9)$

$$\frac{H(10)-H(9)}{1}, \quad \underbrace{\frac{3H(10)-2H(9)-H(10)}{2}}_{\parallel}, \quad \frac{3H(10)-2H(9)-H(9)}{3}$$

$$\underbrace{\frac{2(H(10)-H(9))}{2}}$$

$$\underbrace{H(10)-H(9) = H(10)-H(9) = H(10)-H(9)}$$

$$\Downarrow$$

H is affine linear

$$\Downarrow$$

$$H(S_1) = aS_1 + b$$

$$\Downarrow$$

Multiples of forwards, investments in the risk-free asset and/or the risky asset.

2) Doob-decomposition

Finite probability space

(X_n) - supermartingale has the decomposition

$$X_n = X_0 + M_n - A_n$$

- (M_n) - martingale, $M_0 = 0$
- (A_n) - nondecreasing predictable process, $A_0 = 0$

$$X_n = X_0 + \sum_{k=1}^{n} \Delta X_k$$

$$= X_0 + \underbrace{\sum_{k=1}^{n} E(\Delta X_k \mid \mathcal{F}_{k-1})}_{(\hat{A}_n)\text{-predictable with } \hat{A}_0 = 0} + \underbrace{\sum_{k=1}^{n} (\Delta X_k - E(\Delta X_k \mid \mathcal{F}_{k-1}))}_{(M_n)^{(*)}\text{-martingale with } M_0 = 0}$$

$$(*)\ E(M_{n+1} \mid \mathcal{F}_n) = E\left(\sum_{k=1}^{n+1} (\Delta X_k - E(\Delta X_k \mid \mathcal{F}_{k-1})) \,\Big|\, \mathcal{F}_n\right)$$

$$= \underbrace{\sum_{k=1}^{n} (\Delta X_k - E(\Delta X_k \mid \mathcal{F}_{k-1}))}_{\mathcal{F}_n\text{-measurable}} + \underbrace{E(\Delta X_{n+1} - E(\Delta X_{n+1} \mid \mathcal{F}_n) \mid \mathcal{F}_n)}_{=0}$$

$$= M_n \qquad \forall n$$

If $X_n = X_0 + M_n' + \hat{A}_n'$ holds for another decomposition of "the same type" we have

$$X_n - X_0 = \hat{A}_n' + M_n' = \hat{A}_n + M_n$$

$$X_{n+1} - X_0 = \hat{A}_{n+1}' + M_{n+1}' = \hat{A}_{n+1} + M_{n+1}$$

Hence,

$$\hat{A}'_{n+1} - \hat{A}'_n = M_{n+1} - M_n - (M'_{n+1} - M'_n) + (\hat{A}_{n+1} - \hat{A}_n)$$

$$\Downarrow \; E(\dots | \mathcal{F}_n)$$

$$E(\underbrace{\hat{A}'_{n+1} - \hat{A}'_n}_{\text{measurable}} | \mathcal{F}_n) = \underbrace{E(M_{n+1} - M_n | \mathcal{F}_n)}_{=0} - \underbrace{E(M'_{n+1} - M'_n | \mathcal{F}_n)}_{=0}$$

martingales

$$+ E(\underbrace{\hat{A}_{n+1} - \hat{A}_n}_{\text{measurable}} | \mathcal{F}_n)$$

Hence,

$$\hat{A}'_{n+1} - \hat{A}'_n = \hat{A}_{n+1} - \hat{A}_n. \text{ Since } \hat{A}'_0 = \hat{A}_0 = 0$$

$\Downarrow$ inductively

$$\hat{A}'_n = \hat{A}_n \quad \forall n$$

$$M'_{n+1} - M'_n = M_{n+1} - M_n \text{ with } M'_0 = M_0 = 0$$

$\Downarrow$ inductively

$$M'_n = M_n \quad \forall n$$

$$X_n = X_0 + \hat{A}_n + M_n \qquad \forall n$$

Hence, (M_n)-martingale, $(\hat{A}_n)$-predictable, where the decomposition is unique.

(X_n) - supermartingale

$$\Downarrow$$

$$X_{n+1} - X_n = M_{n+1} - M_n + (\hat{A}_{n+1} - \hat{A}_n)$$

$$\Downarrow \; E(\dots / \mathcal{F}_n)$$

$$\underbrace{E(X_{n+1} - X_n / \mathcal{F}_n)}_{\parallel} = \underbrace{E(M_{n+1} - M_n / \mathcal{F}_n)}_{=0 \text{ martingale}} + E(\underbrace{\hat{A}_{n+1} - \hat{A}_n}_{\text{measurable}} / \mathcal{F}_n)$$

$$0 \geq E(X_{n+1} / \mathcal{F}_n) - X_n = \hat{A}_{n+1} - \hat{A}_n \qquad \forall n$$

(X_n) - supermart.

$$\Downarrow$$

$\hat{A}_{n+1} \leq \hat{A}_n \Rightarrow (\hat{A}_n)$ - non-increasing

$$\Downarrow$$

$(A_n) = (-\hat{A}_n)$ - non-decreasing

with

$X_n = X_0 + M_n - A_n$ - i.e. with the desired properties

3) C_n^A - value at time n of an American option described by (Z_n)

C_n - value at time n of the corresponding European option, defined by $H = Z_N$

3.1) Show $C_n^A \geq C_n$:

Clearly, by definition

$C_N^A = C_N$.

For every n we consider the Snell envelope of (Z_n). Hence, for

$$C_{n+1}^A \overset{(*)}{\geq} C_{n+1}$$

$$C_n^A = \max\left(Z_n, \frac{E_Q(C_{n+1}^A \mid \mathcal{F}_n)}{1+r}\right) \geq \frac{E_Q(C_{n+1}^A \mid \mathcal{F}_n)}{1+r}$$

$$\overset{(*)}{\geq} \frac{E_Q(C_{n+1} \mid \mathcal{F}_n)}{1+r} = C_n$$

3.2) Show that if $Z_n \le C_n$ for any $n \Rightarrow C_n^A = C_n \ \forall n$

We have

- $\tilde{C}_n^A \ge \tilde{Z}_n \ \forall n$ · Prop. 1, Ch. 2, property of the Snell env.
- $\tilde{C}_n \ge \tilde{Z}_n \ \forall n$ - assumption

Hence,

- $(\tilde{C}_n)$-martingale that dominates $(\tilde{Z}_n)$
- martingales are supermartingales
- $(\tilde{C}_n^A)$-smallest supermartingale that dominates $(\tilde{Z}_n)$ (Prop. 1, Ch. 2)

$$\tilde{C}_n^A \le \tilde{C}_n \Rightarrow C_n^A \le C_n \ \forall n$$

From $\tilde{C}_n^A \ge \tilde{C}_n \ \forall n$ in all cases

$\tilde{C}_n^A \le \tilde{C}_n \ \forall n$ for $(\tilde{C}_n) \ge (\tilde{Z}_n)$

$$\tilde{C}_n^A = \tilde{C}_n \ \forall n \text{ for } \tilde{C}_n \ge \tilde{Z}_n \ \forall n$$

Hence,

$C_n^A = C_n \ \forall n$ holds for $C_n \geq Z_n$

3.3) From 3.2 we have to show that $C_n(K, S_n) \geq (S_n - K)_+ = Z_n$ in order to see that $C_n^A = C_n$

- $C_n(K,S_n) = \frac{1}{(1+r)^{N-n}} E_Q((S_N - K)_+ / \mathcal{F}_n)$

$$= (1+r)^n E_Q((\tilde{S}_N - \tilde{K})_+ / \mathcal{F}_n)$$

$$\geq (1+r)^n E_Q((\tilde{S}_N - \tilde{K}) / \mathcal{F}_n)$$

$$= (1+r)^n \tilde{S}_n - \frac{K}{(1+r)^{N-n}}$$

$$= S_n - \frac{K}{(1+r)^{N-n}} \underset{r>0}{\geq} S_n - K$$

- $C_n(K, S_n) \geq 0$

$C_n(K,S_n) \geq Z_n \ \forall n$

$\Downarrow$

$\underline{C_n^A = C_n \ \forall n}$