Exercises; extra material W3

1) Binomial tree

$$H = (S_N - K)_+ = (S_N - 50)_+$$

$$\Delta t = 1/12, \quad N = 4, \quad T = N\Delta t = 1/3 = 4/12$$

$$r_c = 0.2$$

$$u = 1 + b = 1.1 \qquad d = 1 + a = 0.9$$

$$p^* = \frac{u - e^{r_c \Delta t}}{u - d} = \dots$$

$$c(n, x) = e^{-r_c \Delta t}\left((1 - p^*) c(n+1, xu) + p^* c(n+1, xd)\right)$$

e.g.

$$e^{-0.2 \cdot \frac{1}{12}}\left((1 - 0.4159\ldots) \cdot 23.205 + 0.4159\ldots \cdot 9.895\right)$$

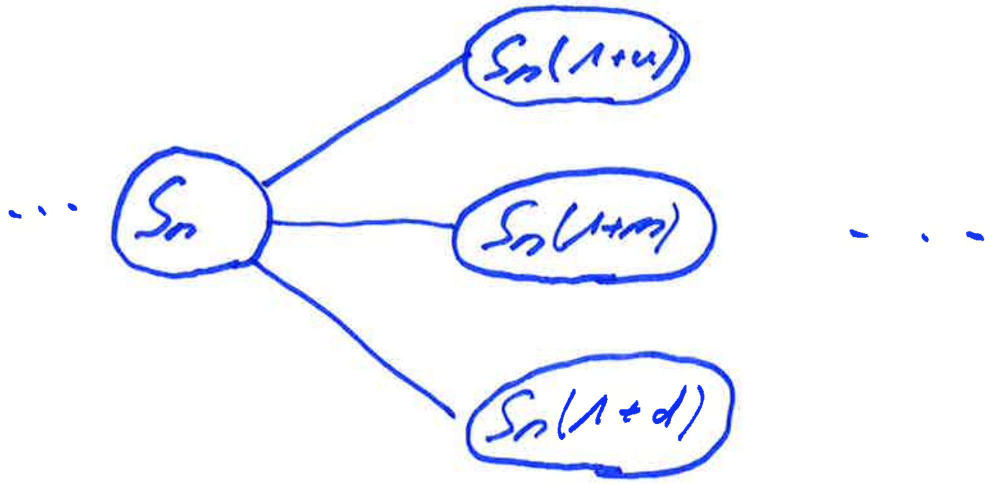
$$\parallel$$

$$17.3764$$

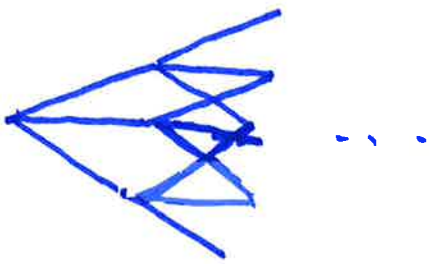
etc.

Rest: See further extra material and solutions.

2)



Note that
$$(1+d)(1+m) = (1+m)(1+d)$$
$$(1+d)(1+u) = (1+u)(1+d)$$
$$(1+m)(1+u) = (1+u)(1+m)$$



Where all paths have strictly positive probability

$\Downarrow$

$0 < \alpha, \beta, \gamma < 1$

For
$$\alpha = Q(T_i = 1+d)$$
$$\beta = Q(T_i = 1+m)$$
$$\gamma = Q(T_i = 1+u)$$

$$T_i = \frac{S_i}{S_{i-1}}$$

$\omega = (\omega_1, \dots, \omega_N)$ where ω_i can be $1+d$, $1+m$, $1+u$

2.1) Show that $r \in (d,u)$ is necessary for a viable market

$0<\alpha,\beta,\gamma<1$ in order to ensure that each path is possible

$\alpha+\beta+\gamma=1$ is needed in order to end up with a probability measure

$(\tilde{S}_n)$ - martingale under Q is also needed

$$E_Q(\tilde{S}_n | \mathcal{F}_{n-1}) = \frac{1}{(1+r)^n} E_Q\left(S_0 \frac{S_1}{S_0} \cdot \frac{S_2}{S_1} \cdots \frac{S_n}{S_{n-1}} \Big| \mathcal{F}_{n-1}\right)$$

$$= \frac{1}{(1+r)^n} E_Q\left(S_0 \prod_{i=1}^{n} T_i \Big| \mathcal{F}_{n-1}\right)$$

$$= \frac{S_0 \prod_{i=1}^{n-1} T_i}{(1+r)^n} E_Q(T_n | \mathcal{F}_{n-1})$$

$$\overset{!}{=} \tilde{S}_{n-1} = \frac{S_0 \prod_{i=1}^{n-1} T_i}{(1+r)^{n-1}}$$

$$\Updownarrow$$

$$\frac{E_Q(T_n | \mathcal{F}_{n-1})}{1+r} = 1$$

Hence, we need

$$E_Q(T_n / \mathcal{F}_{n-1}) = 1+r$$

where

$$E_Q(\underbrace{T_n / \mathcal{F}_{n-1}}_{\text{independent}}) = E_Q(T_n)$$

$$\Downarrow$$

$$1+d < E_Q(T_n) < 1+u \quad \forall n$$

(T_i iid)

$$\Downarrow$$

$$1+d < 1+r < 1+u$$

2.2) $E_Q(T_1) = \alpha(1+d) + \beta(1+m) + \gamma(1+u) = 1+r$

$$\Updownarrow$$

$$\alpha \frac{1+d}{1+r} + \beta \frac{1+m}{1+r} + \gamma \frac{1+u}{1+r} = 1$$

$$\Updownarrow$$

$$\frac{\overbrace{(\alpha+\beta+\gamma)}^{1} + \alpha d + \beta m + \gamma u}{1+r} - \overset{1}{\frac{r+1}{r+1}} = 0$$

$$\Updownarrow$$

$$\frac{\alpha d + \beta m + \gamma u - (\alpha+\beta+\gamma)\cdot r}{1+r} = 0$$

$$\Updownarrow$$

$$\alpha(d-r) + \beta(m-r) + \gamma(u-r) = 0$$

Hence, we have

$$\begin{cases} \alpha+\beta+\gamma = 1 \\ \alpha(d-r)+\beta(m-r)+\gamma(u-r) = 0 \\ 0<\alpha, \beta, \gamma < 1 \end{cases}$$

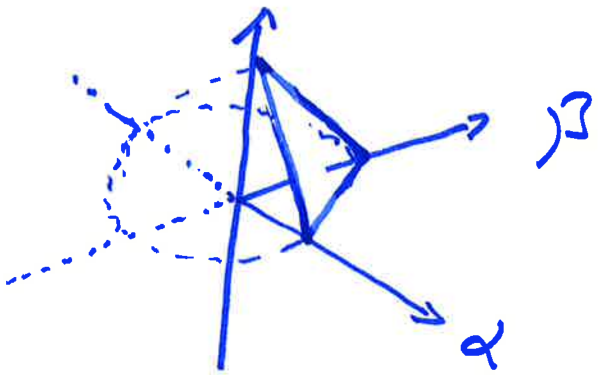
2.3) $\alpha+\beta+\gamma = 1$ intersected with $\mathbb{R}^3_+$ describes the unite simplex

$$\alpha\underset{\overset{\wedge}{0}}{(d-r)} + \beta(m-r) + \gamma\underset{\overset{\vee}{0}}{(u-r)} = 0$$

for a viable market

$\Downarrow$

will intersect the unite-simplex at many points



3) Incomplete markets

3.1) Assume h - attainable, show

$$V_n = S_n^0 E^*\left(\frac{h}{S_N^0} \Big| \mathcal{F}_n\right)$$

where V_n does not depend on $*$

Since h is attainable

$\Downarrow$

$\exists \phi$ s.t.

$$V_n(\phi) = V_0(\phi) + \sum_{i=1}^{n} \phi_i \cdot \Delta S_i \quad \text{and}$$

$$V_N = h \quad \text{a.s.}$$

$\Downarrow$

$$\tilde{V}_N = \tilde{h}$$

$\Updownarrow$

$$\frac{V_N}{S_N^0} = \frac{h}{S_N^0}$$

Hence,

$$E^*\left(\frac{h}{S_N^0} \Big| \mathcal{F}_n\right) = E^*\left(\frac{U_N}{S_N^0} \Big| \mathcal{F}_n\right) = \tilde{U}_n$$

martingale transform

where

$$\tilde{U}_n(\phi) = U_0(\phi) + \sum_{i=1}^{n} \phi_i \Delta \tilde{S}_i$$ "not depending on Q"

3.2) Trinomial model is not complete, however, forwards or riskless investments are still attainable.