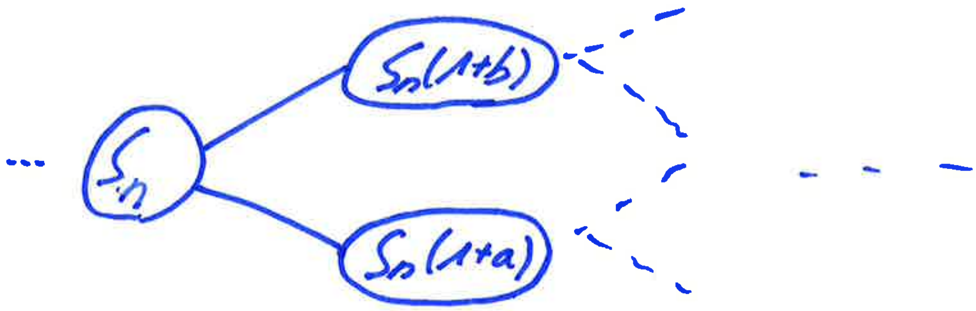
Exercises: Extra material W2

1)



1.1) By the FTAP we have that a market is viable $\Leftrightarrow$ $\exists Q$ equivalent to P such that $E_Q(\tilde{S}_{n+1}|\mathcal{F}_n) = \tilde{S}_n \; \forall n$

Rewrite

$$E_Q(\tilde{S}_{n+1}|\mathcal{F}_n) = \tilde{S}_n$$

$$\Updownarrow$$

$$E_Q\left(\frac{\tilde{S}_{n+1}}{\tilde{S}_n}\Big|\mathcal{F}_n\right) = 1$$

$$\Updownarrow$$

$$E_Q\left(\frac{S_{n+1}(1+r)^n}{S_n(1+r)^{n+1}}\Big|\mathcal{F}_n\right) = 1$$

$$\frac{S_{n+1}}{S_n} = T_{n+1}$$

$$\Updownarrow$$

$$E_Q(T_{n+1}(1+r)^{-1}|\mathcal{F}_n) = 1$$

Thus,

$$E_Q(T_{n+1} | \mathcal{F}_n) = 1+r$$

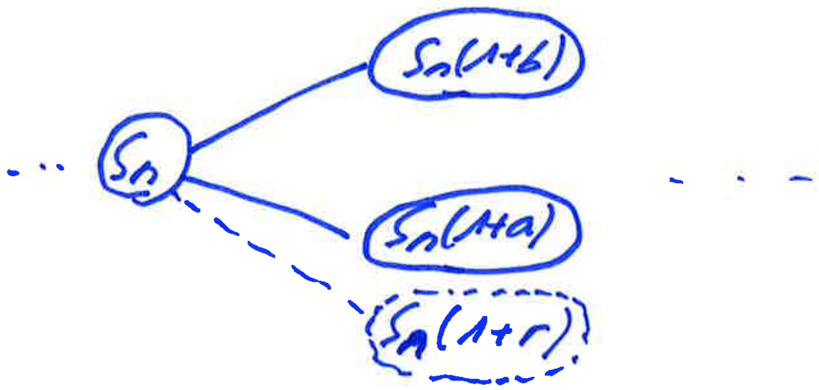
$$\Downarrow \quad \longleftarrow$$

$E(T_{n+1}\mathbb{1}_A) = E((1+r)\mathbb{1}_A) \ \forall A \in \mathcal{F}_n$
in particular
for $\emptyset, \Omega$
$\Downarrow$
$E(T_{n+1} | \mathcal{F}_0) = E(T_{n+1})$
$= 1+r$

$$E_Q(T_{n+1}) = 1+r$$

Since $T_i \in \{1+a, 1+b\}$
with non-zero probability,
we necessarily have $r \in (a,b)$

1,2) $r \leq a$



	Value	CF
1) Borrow some money	$-S_0$	$+S_0$
2) Buy the "cheap" asset	$+S_0$	$-S_0$
V_0	0	0

Buy and hold

$$V_N = S_N - S_0(1+r)^N$$

$$\geq S_0(1+a)^N - S_0(1+r)^N = S_0\big(\overbrace{(1+a)^N}^{\text{always down}} - (1+r)^N\big) \geq 0$$

This works also for $n \in \{0, \dots, N-1\} \Rightarrow V_n \geq 0 \ \forall n$

Furthermore, $\{S_N = S_0(1+b)^N\}$ etc. have strictly positive probabilities, where

$$S_0\big((1+\overset{a}{\hat{b}})^N - (1+r)^N\big) > 0$$

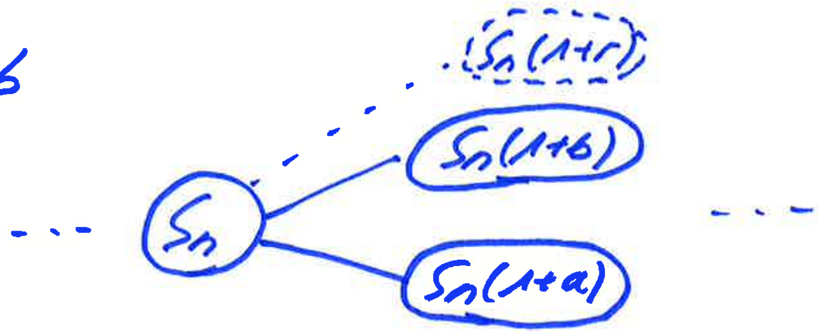
$$\Downarrow$$

$$P(V_N > 0) > 0$$

$$\Downarrow$$

arbitrage

$r \geq b$



	Value	CF
1) Sell the badly performing asset	$-S_0$	S_0
2) Pay into an account	S_0	$-S_0$
V_0	0	0

Buy and hold

$$V_N = S_0(1+r)^N - S_N$$
$$\geq S_0(1+r)^N - S_0(1+b)^N = S_0\left((1+r)^N - \overbrace{(1+b)^N}^{\text{always up}}\right) \geq 0$$

This works also for $n \in \{0, \dots, N-1\}$

$\Downarrow$

$V_n \geq 0 \ \forall n$

Furthermore, $\{S_N = S_0(1+a)^N\}$ etc. have strictly positive probabilities, where $S_0((1+r)^N - (1+a)^N) > 0$ is a strictly positive outcome

$\Downarrow$

$P(V_n > 0) > 0 \implies$ arbitrage

1.3) $r \in (a,b)$, $p^* = \dfrac{b-r}{b-a}$

$(\tilde{S}_n)$ - Q - martingale $\Longleftrightarrow$ $T_1, \dots, T_N$ iid with

$$Q(T_1 = 1+a) = p^*$$
$$= 1 - Q(T_1 = 1+b)$$
$$\left(T_i = \frac{S_i}{S_{i-1}}\right)$$

"$\Longleftarrow$" T_i iid Bernoulli, $S_n = S_0 T_1 \cdots T_n$

$\Downarrow$

$$E_Q(T_{n+1} | \mathcal{F}_n) = E_Q(T_{n+1})$$

$$= (1+a)p^* + (1+b)(1-p^*)$$

$$= (1+a)\frac{b-r}{b-a} + (1+b)\left(1 - \frac{b-r}{b-a}\right)$$

$$= \frac{(1+a)(b-r) + (1+b)(b-a) - (1+b)(b-r)}{b-a}$$

$$= \frac{\cancel{b} - r + \cancel{ab} - ar + b - a + \cancel{b^2} - ab - \cancel{b} + r - \cancel{b^2} + br}{b-a}$$

$$= \frac{r(b-a) + b - a}{b-a} = 1+r$$

$\Downarrow$

$$E_Q\left(\frac{S_{n+1}}{S_n} \middle| \mathcal{F}_n\right) = 1+r$$

$\Downarrow$

$$E_Q\left(\frac{S_{n+1}}{(1+r)^{n+1}} \frac{(1+r)^n}{S_n} \middle| \mathcal{F}_n\right) = 1 \Rightarrow E_Q(\tilde{S}_{n+1} | \mathcal{F}_n) = \tilde{S}_n$$

"$\Longrightarrow$" If for $n = 0, 1, \dots, N-1$

$E_Q(T_{n+1} | \mathcal{F}_n) = 1+r$ we can write

$$1+r = E_Q(T_{n+1}|\mathcal{F}_n) = (1+a)E_Q(1_{\{T_{n+1}=1+a\}}|\mathcal{F}_n) + (1+b)E_Q(1_{\{T_{n+1}=1+b\}}|\mathcal{F}_n)$$

Note that

$$1 = \underbrace{E_Q(1_{\{T_{n+1}=1+a\}}|\mathcal{F}_n)}_{\substack{\| \\ Q(T_{n+1}=1+a|\mathcal{F}_n)}} + \underbrace{E_Q(1_{\{T_{n+1}=1+b\}}|\mathcal{F}_n)}_{\substack{\| \\ Q(T_{n+1}=1+b|\mathcal{F}_n)}}$$

$$1+r = (1+a)\underbrace{Q(T_{n+1}=1+a|\mathcal{F}_n)}_{\substack{\| - \text{notation} \\ p^*}} + (1+b)(1-Q(T_{n+1}=1+a|\mathcal{F}_n))$$

$$\Updownarrow$$

$$(1+r)-(1+b) = p^*(1+a-(1+b))$$

$$\Updownarrow$$

$$\frac{r-b}{a-b} = p^*$$

being constant and does not depend on n for all n.

Note that

$$Q(T_1=x_1, T_2=x_2, \dots, T_n=x_n) = 1\cdot 1\cdot \dots \cdot 1\, Q(T_1=x_1, \dots, T_n=x_n)$$

$$= \frac{Q(T_1=x_1)}{Q(T_1=x_1)} \cdot \frac{Q(T_1=x_1, T_2=x_2)}{Q(T_1=x_1, T_2=x_2)} \cdots \frac{Q(T_1=x_1, T_2=x_2, \dots, T_{n-1}=x_{n-1})}{Q(T_1=x_1, T_2=x_2, \dots, T_{n-1}=x_{n-1})} Q(T_1=x_1, \dots, T_n=x_n)$$

$$= Q(T_1=x_1) Q(T_2=x_2 \mid T_1=x_1) \dots Q(T_n=x_n \mid T_1=x_1, \dots, T_{n-1}=x_{n-1})$$

$$= \prod_{i=1}^{n} p_i$$

p^* for $x_i = 1+a$

$1-p^*$ for $x_i = 1+b$

$\Downarrow$

T_i iid under Q with

$Q(T_i = 1+a) = p^*$

1.4) Hence, we have shown that "$(\tilde{S}_n)$ is a Q-martingale" uniquely determines the distribution of the N-tuple $(T_1, \dots, T_N)$ under Q, and so also the measure Q itself.

$\Downarrow$ FTAP 2

the market is complete

Hence,

$$p(n,H) = \frac{1}{(1+r)^{N-n}} E_Q(H \mid \mathfrak{F}_n)$$

yields the <u>unique</u> arbitrage-free price for the option defined by H.

2.1) Full details in standard solutions

2.2) Recall: Proposition: Let X and Y be two random variables with values in $(E,\mathcal{E})$ and $(F,\mathfrak{F})$, respectively. Assume that X is $\mathcal{B}$-measurable and that Y is independent of $\mathcal{B}$. Then, for any nonnegative (or bounded) Borel function Φ on $(E \times F, \mathcal{E} \otimes \mathfrak{F})$, the function φ defined by

$$\varphi(x) = E(\Phi(x,Y)), \quad x \in E,$$

is a Borel function on $(E,\mathcal{E})$ and we have, with probability one,

$$E(\Phi(X,Y) \mid \mathcal{B}) = \varphi(X).$$

In other words, under the previous assumptions, we can compute $E(\Phi(X,Y) \mid \mathcal{B})$ as if X were a constant.

2.2) With this proposition we can write

$$C_n = \frac{1}{(1+r)^{N-n}} E_Q\Big(\big(\underbrace{S_n}_{\text{measurable}} \underbrace{\prod_{i=n+1}^{N} T_i}_{\text{independent}} - k\big)_+ \,\Big|\, \mathcal{F}_n\Big)$$

as

$C(n, S_n)$ for

$$C(n,x) = \frac{E_Q\big((x \prod_{i=n+1}^{N} T_i - k)_+\big)}{(1+r)^{N-n}}$$

$$= \frac{1}{(1+r)^{N-n}} \sum_{j=0}^{N-n} \frac{(N-n)!}{(N-n-j)!\,j!} (p^*)^j (1-p^*)^{N-n-j} \big(x(1+a)^j(1+b)^{N-n-j} - k\big)_+$$

2.3) Show

$$c(n,x) = \frac{p^*}{1+r}\, c(n+1, x(1+a)) + \frac{1-p^*}{1+r}\, c(n+1, x(1+b))$$

Start with

$$c(n,x) = \frac{E_Q\left(x \prod_{i=n+1}^{N} T_i - k\right)_+}{(1+r)^{N-n}}$$

$$\mathcal{B} = \overline{\sigma(T_{n+2}, \dots, T_N)}$$

$$= \frac{E_Q\left(E_Q\left((x T_{n+1} \cdot T_{n+2} \cdot \dots \cdot T_N - k)_+ \mid \sigma(T_{n+2}, \dots, T_N)\right)\right)}{(1+r)^{N-n}}$$

consider,

independent

$$E_Q\big((x T_{n+1} \underbrace{(T_{n+2} \cdots T_N)}_{\tilde{X}} - k)_+ \mid \mathcal{B}\big) = \varphi(\tilde{X})$$

measurable

for

$$\varphi(\tilde{x}) = E_Q(x T_{n+1} \tilde{x} - k)_+$$

$$= p^*(x(1+a)\tilde{x} - k)_+ + (1-p^*)(x(1+b)\tilde{x} - k)_+$$

$$\Downarrow$$

$$\varphi(\tilde{X}) = E_Q((x T_{n+1} \tilde{X} - k)_+ \mid \mathcal{B})$$

$$= p^*(x(1+a) T_{n+2} \cdots T_N - k)_+ + (1-p^*)(x(1+b) T_{n+2} \cdots T_N - k)_+$$

Hence,

$$c(n,x) = \frac{E_Q\left(p^*(x(1+a)T_{n+2}\cdots T_N - k)_+ + (1-p^*)(x(1+b)T_{n+2}\cdots T_N - k)_+\right)}{(1+r)^{N-n}}$$

$$= \frac{p^* E_Q\left((x(1+a)T_{n+2}\cdots T_N - k)_+\right)}{(1+r)^{N\underbrace{-n-1}_{-(n+1)}}(1+r)} + \frac{(1-p^*) E_Q\left((x(1+b)T_{n+2}\cdots T_N - k)_+\right)}{(1+r)^{N-(n+1)+1}}$$

$$= \frac{p^*}{1+r} c(n+1, x(1+a)) + \frac{1-p^*}{1+r} c(n+1, x(1+b))$$

2.4) We denote the hedging strategy here by ϕ_n^0 and ϕ_n and want to ensure

$$\phi_n^0(1+r)^n + \phi_n S_n = c(n, S_n)$$

Hence,

$$(\mathrm{I}) \quad \phi_n^0 (1+r)^n + \phi_n S_{n-1}(1+a) = c(n, S_{n-1}(1+a))$$

$$(\mathrm{II}) \quad \phi_n^0 (1+r)^n + \phi_n S_{n-1}(1+b) = c(n, S_{n-1}(1+b))$$

$$\Big\Downarrow \; (\mathrm{II}) - (\mathrm{I}) \,/\, S_{n-1}(1+b-(1+a))$$

$$\Delta_n = \phi_n = \frac{c(n, S_{n-1}(1+b)) - c(n, S_{n-1}(1+a))}{S_{n-1}(b-a)}$$

$$\Downarrow$$

$$\underline{\Delta(n,x)} = \underline{\frac{c(n, x(1+b)) - c(n, x(1+a))}{x(b-a)}}$$

Where, as in the proof of Proposition 2

$$\underline{\phi_n^0} = \underbrace{V_0 + \sum_{j=1}^{n-1} \phi_j \Delta \tilde{S}_j}_{\tilde{V}_{n-1}(\phi)} - \phi_n (\tilde{S}_{n-1})$$

complete market

$$\underline{= E_Q\left(\frac{(S_N-h)_+}{(1+r)^N} \Big/ \mathcal{F}_{n-1}\right) - \phi_n \tilde{S}_{n-1}}$$

$$n = 1, \ldots, N$$