Exercises: Extra material WA ①

1) · S_t - price process with $t \in \{0,T\}$

· R_T - risk-free interest rate for $[0,T]$.

Remark: Often other conventions are used.

Here, the initial investment changes by the factor:

· $(1+R_T)$ (over the interval $[0,T]$)

Other conventions are e.g.

· $(1+\hat{R}_T T) = (1+R_T)$ (simply compounded)

$\hat{R}_T$ ↑ per annum

· $e^{\hat{R}_T \cdot T} = (1+R_T)$ (continuously compounded)

where

· $\hat{R}_T = \dfrac{R_T}{T}$

· $\hat{R}_T = \dfrac{1}{T}\log(1+R_T)$ etc.

What is the price F_0 to be fixed at $t=0$ for a contract where we have the right and the obligation to buy the asset for F_0 in T if:

a) Date of payment is $t=0$

Suppose (i) $F_0 > S_0$

$\Downarrow$

$\exists\, \varepsilon > 0$ such that

$F_0 = S_0 + \varepsilon$. We implement the strategy:

	Value	CF
$t=0$ · Buy spot (cheap asset)	$+S_0$	$-S_0$
· Sell the "forward"	$-F_0$	$+F_0$
· Pay ε into an account	$+\varepsilon$	$-\varepsilon$
Total value at $t=0$	0	0

Hence,

1) Value of the strategy at $t=0$ is 0.

2) It is predictable
3) It is self-financing

} implemented at $t=0$ based on information available at $t=0$ with no further modification

4) $V_T = S_T - S_T + \mathcal{E}(1+R_T) > 0$

value of the asset

settle the "forward"

sure profit

Related to "self-financing" also note that

$$V_T = \mathcal{E}(1+R_T)$$

$$= \underbrace{(1\cdot S_0 + (-1)F_0 + \mathcal{E})}_{V_0} + 1\cdot(\underbrace{S_T - S_0}_{\Delta S}) + (-1)(\underbrace{S_T - F_0}_{\Delta F})$$

$$+ \mathcal{E}\underbrace{((1+R_T) - 1)}_{\Delta Bond}$$

$$= V_0 + \underbrace{1\cdot\Delta S + (-1)\Delta F + \mathcal{E}\cdot\Delta Bond}_{\text{gain process}}.$$

Thus, the strategy is self-financing with non-negative values and $V_T > 0$.

$\Downarrow$

arbitrage

(ii) If $F_0 < S_0 \Rightarrow \exists\, \varepsilon > 0$ such that $S_0 = F_0 + \varepsilon$.

We implement:

$t = 0$ (e.g. borrow the asset from a pension fund)

	Value	CF
· sell (the expensive) asset spot	$-S_0$	$+S_0$
· Buy the cheap "forward"	$+F_0$	$-F_0$
· Pay ε into an account	$+\varepsilon$	$-\varepsilon$
Total value at $t=0$	0	0

1) Value of the strategy at $t=0$ is 0.

2) It is predictable
3) It is self-financing

implemented at $t=0$ based on information available at $t=0$ with no further modification.

4) $V_T = -S_T + S_T + \varepsilon(1+R_T) > 0.$

short position in the asset

long position in the "forward"

Furthermore, related to "self-financing" note that

$$V_T = \varepsilon(1+R_T)$$

$$= \underbrace{((-1)S_0 + 1\cdot F_0 + \varepsilon\cdot 1)}_{\substack{\| \\ V_0}} + (-1)\underbrace{(S_T - S_0)}_{\Delta S} + (1)\underbrace{(S_T - F_0)}_{\Delta F}$$

$$+ \varepsilon\underbrace{((1+R_T) - 1)}_{\Delta Bond}$$

$$= V_0 + \underbrace{(-1)\Delta S + (1)\Delta F + \varepsilon \Delta Bond}_{\text{gain process}}$$

Thus, the strategy is self-financing with non-negative values with $V_T > 0$.

$$\Downarrow$$

arbitrage

$$\Downarrow \text{ (i), (ii), no-arbitrage}$$

$$\underline{F_0 = S_0}$$

b) What is the price F_T if we pay at T (usual case)

(i) $F_T > S_0(1+R_T)$

Implement the strategy:

		Value	CF
$t=0$	· Buy spot (the cheap) asset	S_0	$-S_0$
	· Borrow the money	$-S_0$	$+S_0$
	· Short the forward	0	0
	V_0	0	0

$$V_T = \underbrace{S_T}_{\text{value of the asset}} \underbrace{- S_0(1+R_T)}_{\text{pay back the credit}} + \underbrace{(F_T - S_T)}_{\text{forward settlement}} = F_T - S_0(1+R_T) > 0.$$

Furthermore, note that

$$V_T = F_T - S_0(1+R_T)$$

$$= \underbrace{(1\cdot S_0 + (-S_0)\cdot 1)}_{V_0} + 1\cdot\underbrace{(S_T - S_0)}_{\Delta S} + (-1)\underbrace{((S_T - F_T) - 0)}_{\Delta F} + (-S_0)\underbrace{((1+R_T) - 1)}_{\Delta \text{Bond}}$$

Thus, the strategy is self-financing with non-negative values and $V_T > 0$.

$\Downarrow$

arbitrage

(ii) $F_T < S_0(1+R_T)$

		Value	CF
$t=0$	· Sell the (expensive) share	$-S_0$	$+S_0$
	· Pay S_0 into an account	$+S_0$	$-S_0$
	· Enter a long position in the forward	0	0
	V_0	0	0

$$V_T = -S_T + S_0(1+R_T) + (S_T - F_T) = \underset{\vee \atop 0}{S_0(1+R_T) - F_T}$$

Note

$$V_T = S_0(1+R_T) - F_T$$

$$= \underbrace{((-1)S_0 + S_0 \cdot 1)}_{\overset{\parallel}{V_0}} + (-1)\underbrace{(S_T - S_0)}_{\Delta S} + (1)\underbrace{((S_T - F_T) - 0)}_{\Delta F}$$

$$+ S_0\underbrace{((1+R_T) - 1)}_{\Delta \text{Bond}}$$

Thus, the strategy is self-financing with non-negative values and $V_T > 0 \Rightarrow$ arbitrage

$\Downarrow$ (i), (ii), no-arbitrage

$$\underline{F_T = S_0(1+R_T)}$$

i.e. costs of the cash and carry strategy.

c) What is the price F_0^d if R_d is a "dividend yield"?

(i) $F_0^d(1+R_T) < S_0(1+R_T-R_d)$

		Value	CF
$t=0$	• Sell the (expensive) asset	$-S_0$	$+S_0$
	• Buy the (cheap) "forward"	$+F_0^d$	$-F_0^d$
	• Pay the difference into an account	$S_0-F_0^d$	$-(S_0-F_0^d)$
	V_0	0	0

$$V_T = \underbrace{-S_T - S_0 R_d}_{\substack{\text{short share} \\ \text{(incl. compensation)}}} + \underbrace{S_T}_{\text{"forward"}} + \underbrace{(S_0-F_0^d)(1+R_T)}_{\text{account}}$$

$$= S_0(1+R_T-R_d) - F_0^d(1+R_T) > 0$$

$$= \underbrace{V_0}_{=0} + (-1)\underbrace{(S_T+S_0R_d-S_0)}_{\Delta \hat{S} \text{ (incl dividend)}} + (1)\underbrace{(S_T-F_0^d)}_{\Delta\text{Forward}}$$

$$+ (S_0-F_0^d)\underbrace{((1+R_T)-1)}_{\Delta\text{Bond}}$$

Thus, the strategy is self-financing with non-negative values and $V_T>0$. $\Rightarrow$ arbitrage.

(ii) $F_0^d(1+R_T) > S_0(1+R_T-R_d)$

		Value	CF
$t=0$	· Buy the cheap asset	S_0	$-S_0$
	· Sell the expensive "forward"	$-F_0^d$	$+F_0^d$
	· Put the difference on an account	$F_0^d-S_0$	$-(F_0^d-S_0)$
	V_0	0	0

$$V_T = \underbrace{S_T+S_0R_d}_{\text{asset with income}} \underbrace{-S_T}_{\text{"forward" settlement}} + \underbrace{(F_0^d-S_0)(1+R_T)}_{\text{account}}$$

$$= F_0^d(1+R_T) - S_0(1+R_T-R_d) > 0$$

Furthermore,

$$V_T = V_0 + \underbrace{(1)(S_T+S_0R_d-S_0)}_{\Delta S} + \underbrace{(-1)(S_T-F_0^d)}_{\Delta F^d} + \underbrace{(F_0^d-S_0)((1+R_T)-1)}_{\Delta \text{Bond}}$$

Hence, a self-financing strategy with non-negative values and $V_T>0$.

$\Downarrow$ (i),(ii) · no-arbitrage

$$F_0^d = S_0\frac{1+R_T-R_d}{1+R_T}$$ PV of the cash and carry strategy

- $R_d<0$ could e.g. be storage or transportation costs.

Extra exercise: (10)

Income given by R_d
Price at T (more common)

$$F_T^d = S_0(1+R_T-R_d)$$ cash and carry strategy

(i) $F_T^d > S_0(1+R_T-R_d)$

	Value	CF
$t=0$ · Buy spot the (cheap) asset	S_0	$-S_0$
· Borrow the money	$-S_0$	$+S_0$
· Short the forward	0	0
V_0	0	0

$$V_T = \underbrace{S_T + S_0R_d}_{\text{asset}} - \underbrace{S_0(1+R_T)}_{\text{pay money back}} + \underbrace{(F_T^d - S_T)}_{\text{short forward}}$$

$$= F_T^d - S_0(1+R_T-R_d) > 0$$

$$= V_0 + (1)\underbrace{(S_T + S_0R_d - S_0)}_{\Delta S} + (-1)S_0\underbrace{((1+R_T)-1)}_{\Delta \text{Bond}} + (-1)\underbrace{((S_T - F_T^d) - 0)}_{\Delta F^d}$$

Thus, self-financing with non-negative values and $V_T > 0 \Rightarrow$ arbitrage

(ii) $F_T^d < S_0(1+R_T-R_d)$

$t=0$

	Value	CF
· Sell the (expansive) share	$-S_0$	$+S_0$
· Pay S_0 into an account	$+S_0$	$-S_0$
· Enter a long position in the forward	0	0
V_0	0	0

$$V_T = \underbrace{-S_T - S_0 R_d}_{\text{settle short position}} + \underbrace{S_0(1+R_T)}_{\text{account}} + \underbrace{(S_T - F_T^d)}_{\text{forward}}$$

$$= S_0(1+R_T-R_d) - F_T^d$$

$$= V_0 + (-1)\underbrace{(S_T + S_0R_d - S_0)}_{\Delta \hat{S}} + S_0\underbrace{((1+R_T)-1)}_{\Delta \text{Bond}} + (1)\underbrace{((S_T - F_T^d) - 0)}_{\Delta F^d}$$

Hence, a self-financing strategy with non-negative values and $V_T > 0$.

$\Downarrow$

arbitrage

$\Downarrow$ (i), (ii), no-arbitrage

$\underline{F_T^d = S_0(1+R_T-R_d)}$

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2) · C - price of an European call with maturity T and strike K

· P - price of an European put with maturity T and strike K

R_T - as before

(a) Show · $\left(S_0 - \frac{K}{1+R_T}\right)_+ \overset{②}{\leq} C \overset{①}{\leq} S_0$

· $\left(\frac{K}{1+R_T} - S_0\right)_+ \overset{④}{\leq} P \overset{③}{\leq} \frac{K}{1+R_T}$

① $C \leq S_0$; if not $\underline{S_0 < C}$

		Value	CF
$t=0$	· Buy the share	S_0	$-S_0$
	· Sell the call	$-C$	$+C$
	· Pay the difference into an account	$C-S_0$	S_0-C
	V_0	0	0

$$V_T = \underbrace{S_T - (S_T-K)_+}_{\substack{\wedge\!| \\ S_T \\ \overline{\quad} \\ \text{IV} \\ 0}} + \underbrace{(C-S_0)}_{\substack{\vee \\ 0}}(1+R_T) > 0$$

$\Downarrow$

arbitrage

$\Downarrow$ no-arbitrage

$\underline{C \leq S_0}$

② a) Since $(S_T - K)_+ \geq 0 \Rightarrow C \geq 0$,

no-arbitrage
(otherwise "buy" the call, put the money you get into an account...)

b) Furthermore, consider the strategy

	Value	CF
$t=0$ · Buy a call	C	$-C$
· Sell the asset	$-S_0$	S_0
· Invest $\frac{K}{1+R_T}$ in a riskless bond	$\frac{K}{1+R_T}$	$-\frac{K}{1+R_T}$
V_0	$C - S_0 + \frac{K}{1+R_T}$	

At $t = T$

$$V_T = (S_T - K)_+ - S_T + K = -\underbrace{\min(S_T - K, 0)}_{\leq 0} \geq 0$$

① $S_T \geq K$: $(S_T - K)_+ - S_T + K = 0 = -\min(S_T - K, 0)$

② $S_T \leq K$: $\underbrace{(S_T - K)_+}_{0} - S_T + K = -(\underbrace{S_T - K}_{\leq 0}) = -\min(S_T - K, 0)$

Hence, $V_0 \geq 0$

$$V_0 = C - S_0 + \frac{K}{1+R_T} \geq 0$$

$$\Updownarrow$$

$$C \geq S_0 - \frac{K}{1+R_T}$$

$$\Downarrow \text{ no-arbitrage a), b)}$$

$$\underline{C \geq \max\left(0, S_0 - \frac{K}{1+R_T}\right)}$$

Short version: Use 2b)

$$\swarrow \text{European Call-Put Parity}$$

$$C = S_0 - \frac{K}{1+R_T} + \underset{\substack{\Downarrow \\ 0}}{P} \geq S_0 - \frac{K}{1+R_T}$$

$C \geq 0$ as before

$$\Downarrow$$

$$\underline{C \geq \max\left(0, S_0 - \frac{K}{1+R_T}\right)}$$

3 $P \leq \frac{K}{1+R_T}$; if not $P > \frac{K}{1+R_T}$

	Value	CF
$t=0$ · Sell the put	$-P$	$+P$
· Pay the money into an account	$+P$	$-P$
V_0	0	0

$$P = \frac{K}{1+R_T} + \underbrace{P - \frac{K}{1+R_T}}$$

$$V_T = \underbrace{-(K-S_T)_+ + K}_{\geq 0} + \underset{>0}{\left(P - \frac{K}{1+R_T}\right)}\underset{>0}{(1+R_T)} > 0$$

$\Downarrow$

arbitrage

$\Downarrow$ no-arbitrage

$$\underline{P \leq \frac{K}{1+R_T}}$$

(4) $\left(\frac{K}{1+R_T} - S_0\right)_+ \leq P$

a) Again obviously $(K-S_T)_+ \geq 0 \Rightarrow P \geq 0$

b) Consider the strategy

		Value	CF
$t=0$	· Buy a put	P	$-P$
	· Buy the asset	S_0	$-S_0$
	· Borrow	$-\frac{K}{1+R_T}$	$\frac{K}{1+R_T}$
	V_0	$P+S_0-\frac{K}{1+R_T}$	

$\Downarrow$

$$V_T = (K-S_T)_+ + S_T - K = \max(S_T, K) - K$$
$$= (S_T - K)_+ \geq 0$$

$\Downarrow$ no-arbitrage

$$P + S_0 - \frac{K}{1+R_T} \geq 0$$

$\Updownarrow$

$$\underline{P \geq \frac{K}{1+R_T} - S_0}$$

Or again via 2b):

$$P = \underset{\substack{\mathbin{\rotatebox{90}{$\geq$}} \\ 0}}{C} + \frac{K}{1+R_T} - S_0 \geq \frac{K}{1+R_T} - S_0$$

$\Downarrow$ a), b), no-arbitrage

$$\underline{\left(\frac{K}{1+R_T} - S_0\right)_+ \leq P}$$

2b) European Call-Put Parity

$$C + \frac{K}{1+R_T} = P + S_0$$

Consider two portfolios (A and B)

A	B
· long call	· long put
· $\frac{K}{1+R_T}$ in a bond	· long asset

at $t = T$

$$\text{Value PF A} = (S_T - K)_+ + K = \max(S_T, K)$$

$$\| $$

$$(K - S_T)_+ + S_T = \text{Value PF B}$$

$\Downarrow$ · no-arbitrage ①

at $t = 0$ $\quad \text{Value PF A} = C + \frac{K}{1+R_T}$

$$\|$$

$$P + S_0 = \text{Value PF B}$$

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① If the two values at $t=0$ are not the same: Buy the cheap one and sell the expensive one and pay the resulting cash into an account.

$\Downarrow$

arbitrage

3) European and American Options

- European: Can be exercised <u>at</u> maturity
- American: Can be exercised <u>until</u> maturity

assume $r > 0$, $S_n^0 = (1+r)^n$

- C_n, C_n^A - call prices at time n (European / American)
- P_n, P_n^A - put prices at time n (European / American)

a) Derive $C_n \leq C_n^A$ (1)

$P_n \leq P_n^A$ (2)

If (1) does not hold:

$C_n > C_n^A$

	Value
At n: · Buy the American	C_n^A
· Sell the European	$-C_n$
· Pay into an account	$C_n - C_n^A$
V_n	0

At $n = N$

$$V_N = \underbrace{(S_N - K)_+ - (S_N - K)_+}_{=0} + (C_n - C_n^A)(1+r)^{N-n} > 0$$

$\overset{\vee}{0}$

We are long in the American, hence, we can wait.

$\Downarrow$ no-arbitrage

$\underline{C_n \leq C_n^A}$, $n \in \{0, \dots, N\}$

If ② does not hold:

$$P_n > P_n^A$$

		Value
At n:	• Buy the American	P_n^A
	• Sell the European	$-P_n$
	• Pay into an Account	$P_n - P_n^A$
	V_n	0

at $n = N$

$$V_N = (K - S_N)_+ - (K - S_N)_+ + \underbrace{(P_n - P_n^A)}_{\leq 0}(1+r)^{N-n} \geq 0$$

we are long in the American, hence, we can wait.

$$\underbrace{\qquad\qquad}_{= 0}$$

$\Downarrow$ no-arbitrage

$$P_n \leq P_n^A, \qquad n \in \{0, \dots, N\}$$

3b) Show that in our case ($r > 0$, no dividends) $C_n = C_n^A$.

Use the European put-call parity, i.e.

$$C_n + \frac{K}{(1+r)^{N-n}} = P_n + S_n$$

Thus,

$$C_n^A \geq C_n = S_n - \frac{K}{(1+r)^{N-n}} + \underset{\overset{\rotatebox{90}{$\leq$}}{0}}{P_n} \geq S_n - \frac{K}{(1+r)^{N-n}}$$

Ex. 3a)

Since $r > 0$ we have furthermore

$$C_n^A \geq S_n - \frac{K}{(1+r)^{N-n}} > S_n - K$$

what we obtain if we exercise (in case of $S_n \geq K$, otherwise we will clearly not exercise)

Thus,

$$C_n^A > S_n - K$$

value of the American option at time n

amount of money the long position would obtain if he would exercise the option (in case $S_n \geq K$, otherwise he will clearly not exercise)

$\Downarrow$

It is better to keep the option (or sell it in the market).

To exercise before maturity is not optimal.

Remark: If you later check the proof of the European put-call parity at time n in Ex. 2,1) in Week 2 you will see that it does not depend on the concrete CRR model, discussed in Week 2.

The same is clearly not true for puts, consider e.g.

$S_0 \approx 0 \Rightarrow$ if we exercise immediately, we obtain $\approx K$, this will be $\approx K(1+r)^N$ at maturity, while the European put at maturity yields

$$(K-S_N)_+ \leq K < K(1+r)^N$$

$$\Downarrow$$

Thus, there is no hope for $P_0^A = P_0$.

4) Consider a discrete time framework as in the lecture with strictly positive price processes. A sequence $(X_n)_{n=0,\dots,N}$ is a numéraire if

- $(X_n)_n$ is adapted
- $X_0 = 1$, $X_i > 0$ for $i \in \{0, \dots, N\}$
- $X_n = V_n(\theta)$ for a self-financing trading strategy θ

Notation: $S_n^X = \frac{S_n}{X_n}$

a) Economical meaning of S_n^X:

S_n^X is the price of the asset in question if $(X_n)_n$ is the standard by which values are measured.

Remark: Can be helpful for simplifications in valuation, e.g. for exchange options, i.e. $H = (S_T^1 - S_T^2)_+$, maturity

or in cases with stochastic interest rates, e.g. (continuous time):

Prices are then often of the form $E_Q\left(\frac{S_t^0}{S_T^0} H \mid \mathcal{F}_t\right)$.

Changing the numéraire can help to reduce the dimensionality of such valuation problems.

Furthermore, there is also a duality theory for FX-markets where the concept of a numéraire is crucial.

b) Show that $Y_n = S_n^i / S_0^i$ defines a numéraire.

- $\left(S_n^i / S_0^i\right)_n$ - is adapted, since the price processes are adapted.
- $Y_0 = S_0^i / S_0^i = 1$
- $V_n(\theta) = Y_n = \underbrace{\frac{1}{S_0^i}}_{\theta} \cdot S_n^i$ - is a buy and hold strategy with strictly positive values

$\Downarrow$

it is clearly a self-financing strategy.

c) Show that the market including the new numéraire is viable iff the market without is so.

- Given that the enlarged market is viable $\Rightarrow \nexists$ arbitrage strategy of the form $(\phi_n^0, \phi_n^1, \ldots, \phi_n^d, \phi_n^{d+1})$, i.e. in particular there is no strategy of the form

$$(\phi_n^0, \phi_n^1, \ldots, \phi_n^d, 0).$$

- Conversely, assume that the original market is free of arbitrage but not the extended market.

Since (X_n) is a numéraire, we have that

$$X_n = V_n(\theta) = \theta_n \cdot S_n = \theta_n^0 S_n^0 + \theta_n^1 S_n^1 + \ldots + \theta_n^d S_n^d.$$

By assumption there is at least one arbitrage strategy

$$\bar{\phi} = (\bar{\phi}^0, \bar{\phi}^1, \ldots, \bar{\phi}^d, \bar{\phi}^{d+1})$$ in the enlarged market, where

$$V_n(\bar\phi) = \bar\phi_n \cdot (S_n, X_n)$$

$$= \bar\phi^0_n S^0_n + \bar\phi^1_n S^1_n + \dots + \bar\phi^d_n S^d_n + \underline{\bar\phi^{d+1}_n X_n}$$

$$\bar\phi^{d+1}_n \left(\theta^0_n S^0_n + \theta^1_n S^1_n + \dots + \theta^d_n S^d_n\right)$$

$$= \left(\bar\phi^0_n + \bar\phi^{d+1}_n \theta^0_n\right) S^0_n + \left(\bar\phi^1_n + \bar\phi^{d+1}_n \theta^1_n\right) S^1_n$$

$$+ \dots + \left(\bar\phi^d_n + \bar\phi^{d+1}_n \theta^d_n\right) S^d_n$$

$$\Downarrow$$

$$(\tilde\phi_n) = \left(\left(\bar\phi^0_n + \bar\phi^{d+1}_n \theta^0_n,\ \bar\phi^1_n + \bar\phi^{d+1}_n \theta^1_n, \dots, \bar\phi^d_n + \bar\phi^{d+1}_n \theta^d_n\right)\right)$$

is an arbitrage strategy in the non-extended market. (!)

d) $(\phi_n)_n$ - $\mathbb{R}^{d+1}$-valued is self-financing iff $V_n^X(\phi) = \frac{V_n(\phi)}{X_n} = V_0(\phi) + \sum_{j=1}^{n} \phi_n^i \cdot \underbrace{(S_j^X - S_{j-1}^X)}_{\Delta S_j^X}$

By definition ϕ is self-financing iff

$$\phi_{j+1} \cdot S_j = \phi_j \cdot S_j \quad \forall j$$

$$\Updownarrow$$

$$\phi_{j+1} \cdot S_j^X = \phi_j \cdot S_j^X \quad (*)$$

$$\Updownarrow$$

$$-\phi_{j+1} \cdot S_j^X = -\phi_j \cdot S_j^X$$

$$\Updownarrow$$

$$\phi_{j+1} \cdot (S_{j+1}^X - S_j^X) = \underbrace{\phi_{j+1} \cdot S_{j+1}^X}_{V_{j+1}^X(\phi)} - \underbrace{\phi_j \cdot S_j^X}_{V_j^X(\phi)}$$

$$\Updownarrow$$

$$V_j^X(\phi) + \phi_{j+1} \cdot (S_{j+1}^X - S_j^X) \overset{(**)}{=} V_{j+1}^X(\phi)$$

Hence, (**) for all j is equivalent to the self-financing property. Note that $V_0^X(\phi) = V_0(\phi)$.

Hence,

$$\underbrace{\underbrace{\underbrace{V_0(\phi) + \phi_1 \cdot (S_1^X - S_0^X)}_{\overset{(**)}{=}\, V_1^X(\phi)} + \phi_2 \cdot (S_2^X - S_1^X)}_{\overset{(**)}{=}\, V_2^X(\phi)} \cdots}$$

$$\underbrace{V_{n-1}^X(\phi) + \phi_n \cdot (S_n^X - S_{n-1}^X)}_{\overset{(**)}{=}\, V_n^X(\phi)}$$

i.e.

$$V_n^X(\phi) = V_0(\phi) + \sum_{j=1}^{n} \phi_j \cdot \underbrace{(S_j^X - S_{j-1}^X)}_{\Delta S_j^X}$$

Conversely,

$$V^X_{n+1}(\phi) = V_0(\phi) + \sum_{j=1}^{n+1} \phi_j \cdot \Delta S^X_j$$
$$-V^X_n(\phi) = -V_0(\phi) - \sum_{j=1}^{n} \phi_j \cdot \Delta S^X_j$$

$$V^X_{n+1}(\phi) - V^X_n(\phi) = \phi_{n+1} \cdot (S^X_{n+1} - S^X_n)$$

$$\Updownarrow$$

$$V^X_{n+1}(\phi) = V^X_n(\phi) + \phi_{n+1} \cdot (S^X_{n+1} - S^X_n)$$

i.e. (**) holds for all n and this is equivalent to the self-financing property.

4e) Show that in a viable market, there is at most one deterministic numéraire.

Consider the two deterministic numéraires X, Y and recall that we do not create arbitrage opportunities if we enlarge the original market with numéraires, see Exercise 4c).

Clearly for deterministic (and strictly positive) X, Y

$\frac{X}{Y}$ - is also a deterministic process.

Furthermore, this process is constant!

- If not, then we have two time points $i < j$ such that either

a) $i < j$ with $X_i^Y < X_j^Y$

b) $i < j$ with $X_i^Y > X_j^Y$

For a) consider the strategy to buy X at time i by short selling Y, i.e.

$$\phi_i^X = 1, \quad \phi_i^Y = \left(-\frac{X_i}{Y_i}\right)$$

- $t = i$: Value $= (1) \cdot X_i + \left(-\frac{X_i}{Y_i}\right) \cdot Y_i = 0$

At

- $t = j$: Value $= (1) X_j + \left(-\frac{X_i}{Y_i}\right) Y_j = \left(\frac{X_j}{Y_j} - \frac{X_i}{Y_i}\right) Y_j$

$$= (X_j^Y - X_i^Y) Y_j > 0$$

$\Downarrow$

arbitrage

For b) consider the strategy

$t=i$: $V_i = (-1)X_i + \left(\frac{X_i}{Y_i}\right)Y_i = 0$

At

$t=j$: $V_j = (-1)X_j + \left(\frac{X_i}{Y_i}\right)Y_j = \left(\frac{X_i}{Y_i} - \frac{X_j}{Y_j}\right)Y_j$

$= (X_i^Y - X_j^Y)Y_j > 0$

$\Downarrow$

arbitrage

$\Downarrow$ a), b)

$X = cY$

$\uparrow$

constant

Since numéraires are normalized, they coincide.