Martingales in Financial Mathematics

The Snell envelope, optimal stopping,

and American options on finite probability spaces

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Chapter 2: The Snell envelope, optimal stopping, and American options

— Stopping time

— The Snell envelope

— Application to American options

Introduction

Recall that an American option can be exercised at any time until maturity. Hence, at any n the buyer of an American option has to decide whether or not he wants to exercise.

Immediate questions:

— When should the long position stop/exercise ?

— How can the short position hedge the exposure ?

— What about arbitrage (and the FTAP) ?

In view of that it is reasonable to define a sequence (Z_n) representing at any n the realised payoff if the option is exercised at n. Examples:

call $Z_n = (S_n - K)^+$

put $Z_n = (K - S_n)^+$

Clearly, if the option is not exercised before N its value U_N at N is Z_N.

Backward induction

The value of the (non-exercised) option at N is known. If we consider the situation at $N-1$, we have two possibilities, namely

Wait: We keep the option and optain Z_N at N.

Exercise: We exercise the option and immediately optain Z_{N-1}.

While the buyer (with the long position) of the option can choose whether or not he wants to exercise the option, the seller (with the short position) has to be ready to settle the option in both cases.

From now on we assume that the market is viable and complete. Hence, there exists (a unique) risk-neutral probability measure $\mathbb{Q}$ under which the discounted asset price processes are martingales. Furthermore, recall that completeness means that claims can be replicated.

In view of that the writer wants the maximum between Z_{N-1} and the value at time $N-1$ of an admissible strategy paying of Z_N at N, i.e. the value of an American option at N is

$$U_{N-1} = \max\left(Z_{N-1},\ S^0_{N-1}\mathbb{E}^{\mathbb{Q}}[\widetilde{Z}_N \,|\, \mathcal{F}_{N-1}]\right) .$$

By backward induction

$$U_{n-1} = \max\left(Z_{n-1},\ S^0_{n-1}\,\mathbb{E}^{\mathbb{Q}}\left[\widetilde{U}_n \,|\, \mathcal{F}_{n-1}\right]\right), \quad n = 1, \ldots, N$$

$(U_N = Z_N)$. The sequence $(\widetilde{U}_n)$ is the Snell envelope of $(\widetilde{Z}_n)$.

IMPORTANT: Note that $\widetilde{U}_n$ and not $\widetilde{Z}_n$ is used in the conditional expectation.

American options and supermartingales

Proposition 1. *The sequence* $(\widetilde{U}_n)$ *is a supermartingale under* $\mathbb{Q}$*. It is the smallest* $\mathbb{Q}$*-supermartingale that dominates the sequence* $(\widetilde{Z}_n)$*, i.e.* $\widetilde{U}_n \geq \widetilde{Z}_n$*,* $\forall n = 0, \ldots, N$*.*

Theorem 1 (Doob-Meyer Decomposition)**.** *Every supermartingale* $(\widetilde{U}_n)$ *has the unique following decomposition*

$$\widetilde{U}_n = \widetilde{M_n} - \widetilde{A}_n \,,$$

where $(\widetilde{M_n})$ is a martingale and $(\widetilde{A}_n)$ is a non-decreasing, predictable process, null at 0.

Stopping time

Definition 1 (Stopping time)**.** *A random variable*

$$\tau : \Omega \to \{0,\, 1,\, \ldots,\, N\}$$

is a stopping time if,

$$\forall n \in \{0,\, 1,\, \ldots,\, N\} \; : \{\tau \leq n\} \in \mathcal{F}_n \, .$$

Remark: In the discrete time case this is equivalent to $\{\tau = n\} \in \mathcal{F}_n$ for all n.

Stopped processes

Definition 2 (A process stopped at a stopping time)**.** *Let (X_n) be a stochastic process (sequence) adapted to the filtration $(\mathcal{F}_n)$ and let τ be a stopping time. The sequence stopped at time τ is defined by*

$$X_n^\tau(\omega) = X_{n \wedge \tau(\omega)}(\omega)$$

i.e. on the set $\{\tau = j\}$ we have

$$X_n^\tau = \begin{cases} X_j & \text{if} \quad j \leq n \\ X_n & \text{if} \quad j > n\,. \end{cases}$$

Proposition 2. *Let (X_n) be an adapted (to the filtration $(\mathcal{F}_n)$) process and τ be a stopping time. The stopped sequence (X_n^τ) is adapted. Moreover if (X_n) is a $(\mathcal{F}_n)$-martingale (a supermartingale, respectively), then (X_n^τ) is also a martingale (a supermartingale, respectively).*

The Snell envelope and optimal stopping

Proposition 3. *The random variable defined by*

$$\nu = \inf\{n \geq 0, \widetilde{U}_n = \widetilde{Z}_n\}$$

is a stopping time. Furthermore, the stopped sequence $(\widetilde{U}_n^\nu)$ *is a martingale.*

Denote by $\mathcal{T}_{n,N}$ the set of all stopping times taking values in $\{n, n+1, \ldots, N\}$.

Corollary 1. *The stopping time* ν *satisfies*

$$\widetilde{U}_0 = \mathbb{E}^{\mathbb{Q}}[\widetilde{Z}_\nu] = \sup_{\tau \in \mathcal{T}_{0,N}} \mathbb{E}^{\mathbb{Q}}[\widetilde{Z}_\tau]\,.$$

Optimal stopping

Definition 3 (Optimal stopping times)**.** *A stopping time τ^* is called optimal for the sequence $(\widetilde{Z}_n)_{0\le n\le N}$ if*

$$\mathbb{E}^{\mathbb{Q}}[\widetilde{Z}_{\tau^*}] = \sup_{\tau\in\mathcal{T}_{0,N}} \mathbb{E}^{\mathbb{Q}}[\widetilde{Z}_\tau]\,.$$

Hence, ν is optimal. The following result gives a characterisation of optimal stopping times that shows that ν is the smallest optimal stopping time.

Proposition 4 (The "smallest" optimal stopping time)**.** *The stopping time τ^* is optimal if and only if*

$$\begin{cases} \widetilde{U}_{\tau^*} = \widetilde{Z}_{\tau^*} \text{ and} \\ (\widetilde{U}_{\tau^*\wedge n}) \text{ is a } (\mathcal{F}_n)\text{-martingale}\,. \end{cases}$$

Proposition 5 (The "largest" optimal stopping time)**.** *The largest optimal stopping time for* $(\widetilde{Z}_n)$ *is given by*

$$\nu_{\max} = \begin{cases} N & \text{if } \widetilde{A}_N = 0 \\ \inf\{n\,,\, \widetilde{A}_{n+1} \neq 0\} & \text{if } \widetilde{A}_N \neq 0\,. \end{cases}$$

Application to American options

Recall that

call $Z_n = (S_n - K)^+$

put $Z_n = (K - S_n)^+$.

We have identified (U_n) as being the value process (sequence) corresponding to an American option, defined by the process (Z_n), where U_n is defined by

$$\begin{cases} U_N & = Z_N \\ U_n & = \max\left(Z_n, S_n^0 \mathbb{E}^{\mathbb{Q}}\left[\frac{U_{n+1}}{S_{n+1}^0} \middle| \mathcal{F}_n\right]\right) \quad \forall n \leq N-1 \, . \end{cases}$$

Thus, the process $(\widetilde{U}_n)$ define by U_n/S_n^0 (≈ the discounted values of the option) is the Snell envelope under $\mathbb{Q}$ of the process $(\widetilde{Z}_n)$. We deduce from Corollary 1 that

$$\widetilde{U}_0 = \sup_{\tau \in \mathcal{T}_{0,N}} \mathbb{E}^{\mathbb{Q}}[\widetilde{Z}_\tau] = \sup_{\tau \in \mathcal{T}_{0,N}} \mathbb{E}^{\mathbb{Q}}\left[\frac{Z_\tau}{S_\tau^0}\right] .$$

In view of Theorem 1 we can write

$$\widetilde{U}_n = \widetilde{M}_n - \widetilde{A}_n ,$$

where $(\widetilde{M}_n)$ is a $\mathbb{Q}$ martingale and $(\widetilde{A}_n)$ is a predictable, non-decreasing process null at 0. Since the market is complete there is a self-financing strategy ϕ such that

$$V_N(\phi) = S_N^0 \widetilde{M}_N .$$

We have

$$\mathbb{E}^{\mathbb{Q}}[\widetilde{V}_N(\phi)|\mathcal{F}_n] = \mathbb{E}^{\mathbb{Q}}[\widetilde{M}_N|\mathcal{F}_n] = \widetilde{M}_n .$$

Hence,

$$\widetilde{U}_n = \widetilde{V}_n(\phi) - \widetilde{A}_n \quad \Leftrightarrow \quad U_n = V_n(\phi) - A_n .$$

Thus, the writer of the option can hedge himself perfectly: Once he receives the premium $U_0 = V_0(\phi)$, he can get wealth equal to $V_n(\phi)$ at time n, which is bigger than U_n and a fortiori Z_n.

Optimal date to exercise the option

What is the optimal date to exercise the option ?

The date has to be chosen among the stopping times.

If $U_n > Z_n$ at n we have that the date is not optimal (we would trade a option worth U_n for an amount of Z_n).

An optimal date τ for exercising the option is given by $U_\tau = Z_\tau$.

Exercising after $\nu_{\max}$ is not optimal since exercising the option yields $U_{\nu_{\max}} = V_{\nu_{\max}}(\phi)$ and following the strategy from that time, with the money obtained by exercising the option, creates a portfolio whose value is strictly bigger than U_n at time $\nu_{\max} + 1, \ldots, N$.

Therefore we set, as a second condition, $\tau \leq \nu_{\max}$, which allows to say that $\tilde{U}^\tau$ *is a* **martingale**.

As a result, optimal dates of exercise are optimal stopping times for the process $(\widetilde{Z}_n)$ under the probability measure $\mathbb{Q}$.

Hedgers point of view: If $U_\tau > Z_\tau$ or $A_\tau > 0$

$$V_\tau(\phi) - Z_\tau = U_\tau + A_\tau - Z_\tau \; ,$$

which is in both cases strictly positive.