# Exercises Martingales in Financial Mathematics: Static and semi-static hedging (Solutions)

Week 9, 2025

We assume a risk-neutral Black–Scholes setting, i.e. we have a risk-less bond with price process $B_t = e^{rt}$, $t \in [0,T]$, $r > 0$ and a risky asset with a price process satisfying

$$dS_t = rS_t dt + \sigma S_t dW_t \,,$$

where $\sigma > 0$ and $(W_t)_{t\in[0,T]}$ is a standard Brownian motion (with respect to $\mathbb{Q}$).

## Exercise 1: Forward price

Recall that a forward contract is an agreement to pay a specified delivery price $k$ at a maturity date $T \geq 0$ for the asset whose price at time $t$ is $S_t$.

The $T$-forward price $F_{t,T}$ of this asset at time $t \in [0,T]$ is the value of $k$ that makes the forward contract have no-arbitrage price zero at time $t$. Describe the process $F_{t,T}$, $t \in [0,T]$.

*For $k$ the payoff at time $T$ will be $(S_T - k) = (S_T - k)_+ - (k - S_T)_+$ where the l.h.s. is the difference of two non-negative (and square-integrable) payoff functions. Hence, we can apply the usual risk-neutral valuation formula (along with linearity of conditional expectations) in order to compute the price at time $t$ of the forward contract for a delivery price $k$*

$$e^{-(T-t)r}\mathbb{E}_{\mathbb{Q}}[(S_T - k)|\mathcal{F}_t] = e^{rt}\mathbb{E}_{\mathbb{Q}}[e^{-rT}S_T|\mathcal{F}_t] - e^{-(T-t)r}k = e^{rt}\tilde{S}_t - e^{-(T-t)r}k = S_t - e^{-(T-t)r}k \,,$$

*where $(\tilde{S}_t)_{t\in[0,T]}$ stands for the discounted asset price process being a $\mathbb{Q}$ martingale. In order for this to be zero, $k$ must be given by*

$$F_{t,T} = S_t e^{r(T-t)} \,, \quad t \in [0,T] \,.$$

*Note that this process satisfies*

$$dF_{t,T} = F_{t,T}\sigma dW_t \,,$$

*under $\mathbb{Q}$.*

## Exercise 2: European put-call symmetry

Show that in the Black–Scholes case the European put-call symmetry holds, which can be expressed by the property that for arbitrary $k \geq 0$

$$\begin{aligned} e^{-r(T-t)}\mathbb{E}_{\mathbb{Q}}[(F_{t,T}\eta_{t,T} - k)_+|\mathcal{F}_t] &= e^{-r(T-t)}\mathbb{E}_{\mathbb{Q}}[(F_{t,T} - k\eta_{t,T})_+|\mathcal{F}_t] \\ &= e^{-r(T-t)}\frac{k}{F_{t,T}}\mathbb{E}_{\mathbb{Q}}\Big[\big(\frac{F_{t,T}^2}{k} - F_{t,T}\eta_{t,T}\big)_+|\mathcal{F}_t\Big] \,, \end{aligned}$$

where $\eta_{t,T} = e^{-\frac{1}{2}\sigma^2(T-t)+\sigma(W_T-W_t)}$, $t \in [0,T]$, so that $F_{t,T}\eta_t = S_T$ under $\mathbb{Q}$, i.e. we obtain a relation between European call and put prices (often this relation is only formulated for $t=0$, i.e. for the random variable $S_T$).
Hint: Write the Balck–Scholes formulas in terms of $F_{t,T}$.

*From the lecture course we know that*

$$C_t = e^{-r(T-t)}\mathbb{E}_{\mathbb{Q}}[(S_T-k)_+|\mathcal{F}_t] = S_t\mathcal{N}(d_+) - ke^{-r(T-t)}\mathcal{N}(d_-)\,,$$
$$P_t = e^{-r(T-t)}\mathbb{E}_{\mathbb{Q}}[(k-S_T)_+|\mathcal{F}_t] = ke^{-r(T-t)}\mathcal{N}(-d_-) - S_t\mathcal{N}(-d_+)\,,$$

*with*

$$d_\pm = \frac{\log\left(\frac{S_t}{k}\right) + \left(r \pm \frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}} = \frac{\log\left(\frac{S_t e^{r(T-t)}}{k}\right) \pm \frac{\sigma^2}{2}(T-t)}{\sigma\sqrt{T-t}}$$
$$= \frac{\log\left(\frac{F_{t,T}}{k}\right) \pm \frac{\sigma^2}{2}(T-t)}{\sigma\sqrt{T-t}}\,.$$

*Hence,*

$$P_t = e^{-r(T-t)}\left(k\mathcal{N}\left(-\frac{\log\left(\frac{F_{t,T}}{k}\right) - \frac{\sigma^2}{2}(T-t)}{\sigma\sqrt{T-t}}\right) - F_{t,T}\mathcal{N}\left(-\frac{\log\left(\frac{F_{t,T}}{k}\right) + \frac{\sigma^2}{2}(T-t)}{\sigma\sqrt{T-t}}\right)\right)$$

*where $F_{t,T}$ is $\mathcal{F}_t$-measurable. On the other hand we have*

$$C_t = e^{-r(T-t)}\left(F_{t,T}\mathcal{N}\left(\frac{\log\left(\frac{F_{t,T}}{k}\right) + \frac{\sigma^2}{2}(T-t)}{\sigma\sqrt{T-t}}\right) - k\mathcal{N}\left(\frac{\log\left(\frac{F_{t,T}}{k}\right) - \frac{\sigma^2}{2}(T-t)}{\sigma\sqrt{T-t}}\right)\right)$$
$$= e^{-r(T-t)}\left(F_{t,T}\mathcal{N}\left(-\frac{\log\left(\frac{k}{F_{t,T}}\right) - \frac{\sigma^2}{2}(T-t)}{\sigma\sqrt{T-t}}\right) - k\mathcal{N}\left(-\frac{\log\left(\frac{k}{F_{t,T}}\right) + \frac{\sigma^2}{2}(T-t)}{\sigma\sqrt{T-t}}\right)\right)\,,$$

*where* $F_{t,T}$ is $\mathcal{F}_t$-measurable, *i.e. we end up with the put formula for a put with "strike" $F_{t,T}$ and a "forward" price $k$, c.f. also the derivation of the Black–Scholes formula, so that we obtain the first equality. The second equality is simply obtained by the positive homogeneity of payoff functions of calls and puts.*

*Note that for $F_{t,T} = k$ the European put-call symmetry coincides with the relation obtained from European put-call parity. However, it is essential to stress that the parity holds for most models (up to some problems related to bubble modelling / problems with strict local martingales) while the put-call symmetry is a much more model dependent property.*

## Exercise 3: Semi-static hedge of a down-and-out call

Semi-static hedging strategies are often defined to be replicating strategies where trading is no more needed than two times after inception. Assume that there is a barrier $H$, a strike $k$ satisfying $H < k$, where $S_0 > H$, and assume that there are no carrying costs. The down-and-out call is knocked-out if $H$ is hit any time before maturity. Otherwise pays $(S_T - k)_+$, i.e.

$$X_{\text{doc}} = (S_T - k)_+ \mathbb{1}_{S_t > H\,,\forall t \in [0,T]}\,.$$

Assume that European calls and puts are available for arbitrary strikes. Use the European put-call symmetry in order to derive a semi-static hedging strategy (use without proof that the put-call symmetry from Exercise 2 also holds for $[0,T]$-valued stopping times).

*Remark: That the European put-call symmetry from Exercise 2 also holds for $[0,T]$-valued stopping times is a consequence of the strong Markov property of Brownian motion.*
*We apply the following replication: Buy a European call with strike $k$ and sell $kH^{-1}$ puts with strike $H^2k^{-1}$.*

- *If the barrier is avoided we have in particular that $S_T > H$, so that $H^2/k - S_T < 0$ (since $H < k$). Hence, the puts expire worthless while the call replicates the contract.*

- *If the barrier is hit we have by the sample path continuity of $(S_t)_{t\in[0,T]}$ that $S_\tau = H$ on $\{\tilde{\tau} \le T\}$, for the stopping time $\tau = \tilde{\tau} \wedge T$, $\tilde{\tau} = \inf\{t : S_t = H\}$. Furthermore, we have for vanishing carrying costs that $S_t = F_{t,T}$ for all $t \in [0,T]$ so that by applying the European put-call symmetry for $[0,T]$-valued stopping times (along with the positive homogeneity of the payoff functions of calls and puts) we see that the long and the short position have the same price at $\tau$ on the event $\{\tilde{\tau} \le T\}$, i.e. we can close our position at zero costs. In formulas we have on $\{\tilde{\tau} \le T\}$ $(r = 0)$*

$$\begin{aligned}\mathbb{E}_{\mathbb{Q}}[(S_\tau \eta_{\tau,T} - k)_+ | \mathcal{F}_\tau] &= \mathbb{E}_{\mathbb{Q}}[(H\eta_{\tau,T} - k)_+ | \mathcal{F}_\tau] \\ &= \mathbb{E}_{\mathbb{Q}}[(H - k\eta_{\tau,T})_+ | \mathcal{F}_\tau] \\ &= \frac{k}{H}\mathbb{E}_{\mathbb{Q}}\Big[\Big(\frac{H^2}{k} - H\eta_{\tau,T}\Big)_+ | \mathcal{F}_\tau\Big] \\ &= \frac{k}{H}\mathbb{E}_{\mathbb{Q}}\Big[\Big(\frac{H^2}{k} - S_\tau\eta_{\tau,T}\Big)_+ | \mathcal{F}_\tau\Big],\end{aligned}$$

*where $\mathcal{F}_\tau = \{A \in \mathcal{F} : A \cap \{\tau \le t\} \in \mathcal{F}_t \text{ for all } t\}$ is the stopping time $\sigma$-algebra; i.e. intuitively $\mathcal{F}_\tau$ represents the events known at time $\tau$.*

*Remark: Semi-static hedges are also known for many other barrier options. However, usually, the hedges are more complicated. Furthermore, unlike in many other situations, generalisations to non-vanishing carrying costs are non-trivial but possible (based on a "quasi-symmetry"). The resulting hedge is then still of the European type but usually needs to be decomposed in more liquidly traded instruments (see Exercise 4 for the corresponding idea).*

# Exercice 4: Decomposition of European payoff functions

Assume that a payoff function $f\colon \mathbb{R}_+ \to \mathbb{R}$ is two times *continuously* differentiable. Show that for $a \in \mathbb{R}_+$

$$f(x) = f(a) + f'(a)(x-a) + \int_a^\infty f''(k)(x-k)_+ dk + \int_0^a f''(k)(k-x)_+ dk\,,$$

and give an economic interpretation.

*In what follows, we use the convention $\int_a^b f(x)dx = -\int_b^a f(x)dx$. Furthermore, recall that for $A \subset \mathbb{R}$, $f\colon A \to \mathbb{R}$ is differentiable of a certain order on $A$ if $f$ can be extended to a* differentiable function of the same order *on an open set $U \supset A$. By the fundamental theorem of calculus, by* *integration by parts, and by the formula* $xf'(x) = \int_a^x xf''(t)dt + xf'(a)$*, we have, for any* $a \in \mathbb{R}_+$

$$\begin{aligned} f(x) &= f(a) + \int_a^x f'(k)dk = f(a) + xf'(x) - af'(a) - \int_a^x kf''(k)dk \\ &= f(a) + \int_a^x xf''(k)dk + xf'(a) - af'(a) - \int_a^x kf''(k)dk \\ &= f(a) + f'(a)(x-a) + \int_a^x f''(k)(x-k)dk \\ &= f(a) + f'(a)(x-a) + \int_a^x f''(k)(x-k)_+dk - \int_a^x f''(k)(k-x)_+dk\,. \end{aligned}$$

*We arrive at*

$$f(x) = f(a) + f'(a)(x-a) + \int_a^\infty f''(k)(x-k)_+dk + \int_0^a f''(k)(k-x)_+dk\,,$$

$x \in \mathbb{R}_+$*. The economical interpretation of this representation is that if we let* $a$ *be the current forward price, we have a static hedge with bonds, forwards, and lots of options (where the options are out of and at the money in a certain sense).*

*Remark: If the integrals are Lebesgue integrals we cannot drop the continuity assumption on the second derivative without assuming that the first derivative is absolutely continuous. However, the representation can be extended to functions which are the difference of two convex functions with existing and finite right derivative at* $0$*, by e.g. applying the Lebesgue-Stieltjes integral (so that the hedges are given by Lebesgue-Stieltjes measures). This representation can even be generalised to functions which are the difference of two continuous (at* $0$ *this is not guaranteed) convex functions, however, the corresponding measures do not necessarily satisfy that they are finite on any compact sets. It can also be shown that the representable payoff functions are e.g. dense in* $L^1$*. However, it is probably worth to mention that in the existing corresponding literature there are many problematic statements as far as boundary anomalies at* $0$ *are concerned.*

# Exercise 5: Implicit distribution

Consider a risk-neutral setting where $\tilde{S}_T$ is sampled from a martingale and assume for simplicity that the strictly positive random variable $S_T$ is absolutely continuous with continuous density $q$. Show that either the prices of European calls or European puts for arbitrary strikes uniquely determine the distribution of $S_T$.[1]

*Consider the non-discounted prices of all European call options with strike* $k > 0$ *and use the existence of the continuous density along with the European put-call parity in order to see that*

$$\mathbb{E}_{\mathbb{Q}}(S_T - k)_+ = S_0e^{rT} - k + \mathbb{E}_{\mathbb{Q}}(k - S_T)_+ = S_0e^{rT} - k + \int_0^k (k-x)q(x)dx\,.$$

*Since the integrand is continuous, we have that this expression is differentiable in* $k$ *(and since continuous functions are integrable over compact sets we can also split the integral in the sum of two integrals) so that by computing the derivative with respect to* $k$ *we obtain*

$$\frac{\partial}{\partial k}(\mathbb{E}_{\mathbb{Q}}(S_T - k)_+) = -1 + \int_0^k q(x)dx + kq(k) - kq(k) = -1 + \int_0^k q(x)dx\,,$$

[1] This result is obviously interesting in its own right but has also some consequences for the caracterisation of random variables satisfying European put-call symmetry.

*being again differentiable in $k$. Hence, by again differentiating with respect to $k$ we end up with*

$$\frac{\partial^2}{\partial k^2}(\mathbb{E}_{\mathbb{Q}}(S_T - k)_+) = q(k)\,,$$

*for arbitrary $k > 0$. Hence, the prices of all European calls (and obviously alternatively the prices of all European puts) determine the density and hence, the law of $S_T$ under $\mathbb{Q}$.*
*Remark: This result also holds in not necessarily absolutely continuous settings. Often this is not proved in the financial literature. A general careful proof (with multivariate extensions) for integrable $S_T$ can e.g. be found (after a slight economic reinterpretation) in K. Mosler, Multivariate Dispersion, Central Regions and Depth. Volume 165 of Lect. Notes Statist., Springer 2002.*