Exercises Martingales in Financial Mathematics: Black–Scholes and Brownian Motion (Solutions)

Week 8, 2025

Recall that in a Black–Scholes setting, the price of a European call is given by

$$V_t = C(T-t,\, S_t,\, K,\, r,\, \sigma) = S_t\, \mathcal{N}(d_+) - K\, e^{-r(T-t)}\, \mathcal{N}(d_-)$$

with

$$d_\pm = \frac{\log \frac{S_t}{K} + \left(r \pm \frac{1}{2}\sigma^2\right)\,(T-t)}{\sigma\sqrt{T-t}}\,.$$

Exercise 1: Black–Scholes model and Greeks

We are interested in the prices of European call and put options. In particular in the partial derivatives of the pricing function with respect to model parameters.

1. Use the European put-call parity in order to (re)derive the pricing formula for European puts.

   The European put-call parity states that $C - P = S_t - K\, e^{-r(T-t)}$. *Hence,*

   $$P(T-t,\, S_t,\, K,\, r,\, \sigma) = K\, e^{-r(T-t)}\, [1 - \mathcal{N}(d_-)] - S_t\, [1 - \mathcal{N}(d_+)]\,.$$

2. Compute the deltas

   $$\begin{aligned} \Delta_C(T-t,\, S_t,\, K,\, r,\, \sigma) &= \frac{\partial C}{\partial x}(T-t,\, x,\, K,\, r,\, \sigma)\Big|_{x=S_t} = \mathcal{N}(d_+) \\ \Delta_P(T-t,\, S_t,\, K,\, r,\, \sigma) &= \frac{\partial P}{\partial x}(T-t,\, x,\, K,\, r,\, \sigma)\Big|_{x=S_t} = \mathcal{N}(d_+) - 1 \end{aligned}$$

3. Compute the gammas

   $$\begin{aligned} \Gamma_C(T-t,\, S_t,\, K,\, r,\, \sigma) &= \frac{\partial^2 C}{\partial x^2}(T-t,\, x,\, K,\, r,\, \sigma)\Big|_{x=S_t} = \frac{1}{\sqrt{2\,\pi\,(T-t)}\, S_t\, \sigma}\, e^{-\frac{1}{2}d_+^2} \\ \Gamma_P(T-t,\, S_t,\, K,\, r,\, \sigma) &= \frac{\partial P^2}{\partial x^2}(T-t,\, x,\, K,\, r,\, \sigma)\Big|_{x=S_t} = \frac{1}{\sqrt{2\,\pi\,(T-t)}\, S_t\, \sigma}\, e^{-\frac{1}{2}d_+^2} \end{aligned}$$

4. Compute the vegas

   $$\begin{aligned} \mathrm{Vega}_C(T-t,\, S_t,\, K,\, r,\, \sigma) &= \frac{\partial C}{\partial x}(T-t,\, S_t,\, K,\, r,\, x)\Big|_{x=\sigma} = S_t\, \frac{1}{\sqrt{2\,\pi}}\, e^{-\frac{1}{2}d_+^2}\, \sqrt{T-t} \\ \mathrm{Vega}_P(T-t,\, S_t,\, K,\, r,\, \sigma) &= \frac{\partial P}{\partial x}(T-t,\, S_t,\, K,\, r,\, x)\Big|_{x=\sigma} = S_t\, \frac{1}{\sqrt{2\,\pi}}\, e^{-\frac{1}{2}d_+^2}\, \sqrt{T-t} \end{aligned}$$

5. Compute the thetas

$$
\begin{aligned}
\Theta_C(T-t,\,S_t,\,K,\,r,\,\sigma) &= -\frac{\partial C}{\partial s}(s,\,S_t,\,K,\,r,\,\sigma)\bigg|_{s=T-t} \\
&= -\frac{S_t\,\sigma}{2\sqrt{2\,\pi\,(T-t)}}\,e^{-\frac{1}{2}d_+^2} - r\,K\,e^{-r(T-t)}\mathcal{N}(d_-) \\
\Theta_P(T-t,\,S_t,\,K,\,r,\,\sigma) &= -\frac{\partial P}{\partial s}(s,\,S_t,\,K,\,r,\,\sigma)\bigg|_{s=T-t} \\
&= r\,K\,e^{-r(T-t)}\,[1-\mathcal{N}(d_-)] - \frac{S_t\,\sigma\,e^{-\frac{1}{2}d_+^2}}{2\sqrt{2\,\pi\,(T-t)}}
\end{aligned}
$$

6. Compute the rhos

$$
\begin{aligned}
\rho_C(T-t,\,S_t,\,K,\,r,\,\sigma) &= \frac{\partial C}{\partial p}(T-t,\,S_t,\,K,\,p,\,\sigma)\bigg|_{p=r} = K\,(T-t)\,e^{-r(T-t)}\mathcal{N}(d_-) \\
\rho_P(T-t,\,S_t,\,K,\,r,\,\sigma) &= \frac{\partial P}{\partial p}(T-t,\,S_t,\,K,\,p,\,\sigma)\bigg|_{p=r} = -K\,(T-t)\,e^{-r(T-t)}\,[1-\mathcal{N}(d_-)]
\end{aligned}
$$

Some details of the calculations. Denote $\tau = T-t$ and $\varphi(d) = \mathcal{N}'(d)$. By slightly abusing the notation it is helpful to show that

$$
\varphi(d_-) = \varphi(d_+)\frac{Se^{r\tau}}{K}\,.
$$

$$
\begin{aligned}
\varphi(d_-) &= \frac{1}{\sqrt{2\pi}}\exp(-\frac{1}{2}(d_+-\sigma\sqrt{\tau})^2) \\
&= \frac{1}{\sqrt{2\pi}}\exp(-\frac{1}{2}\,d_+^2 + \log\left(\frac{S}{K}\right) + r\tau + \frac{1}{2}\,\sigma^2\tau - \frac{1}{2}\sigma^2\tau) \\
&= \varphi(d_+)\frac{S}{K}e^{r\tau}\,.
\end{aligned}
$$

The calculation of the Greeks is now rather straight forward. For example the delta of a European call (and similarly the one of a European put) is obtained by computing

$$
\begin{aligned}
\Delta = \frac{\partial V_t}{\partial S} &= \mathcal{N}(d_+) + S\varphi(d_+)\frac{\partial d_+}{\partial S} - Ke^{-r\tau}\varphi(d_-)\frac{\partial d_-}{\partial S} \\
&= \mathcal{N}(d_+) + S\varphi(d_+)\frac{1}{\sigma\sqrt{\tau}}\frac{K}{S}\frac{1}{K} - Ke^{-r\tau}\varphi(d_-)\frac{1}{\sigma\sqrt{\tau}}\frac{K}{S}\frac{1}{K} \\
&= \mathcal{N}(d_+) + \frac{e^{-r\tau}K}{\sigma\sqrt{\tau}S}\left(\frac{e^{r\tau}S}{K}\varphi(d_+) - \varphi(d_-)\right) \\
&= \mathcal{N}(d_+)\,.
\end{aligned}
$$

Recall that this is the number of underlying instruments we have to buy in order to locally *hedge the European call with the help of the replication portfolio.*

By again slightly abusing the notation we derive the other Greeks as follows

$$
\begin{aligned}
\Gamma &= \frac{\partial^2 V_t}{\partial S^2} = \frac{\partial \Delta}{\partial S} = \frac{\partial}{\partial S}\left(\mathcal{N}(d_+)\right) \\
&= \frac{\varphi(d_+)}{\sigma\sqrt{\tau}S}\,. \\
\mathcal{V} &= \frac{\partial V_t}{\partial \sigma} \\
&= S\varphi(d_+)\left(\frac{\sigma\tau}{\sigma\sqrt{\tau}} - \frac{\log(\frac{S}{K}) + (r+\frac{1}{2}\sigma^2)\tau}{\sigma^2\sqrt{\tau}}\right) \\
&\quad - Ke^{-r\tau}\varphi(d_-)\left(-\frac{\sigma\tau}{\sigma\sqrt{\tau}} - \frac{\log(\frac{S}{K}) + (r-\frac{1}{2}\sigma^2)\tau}{\sigma^2\sqrt{\tau}}\right) \\
&= S\varphi(d_+)\left(\sqrt{\tau} - \frac{\log(\frac{S}{K}) + (r+\frac{1}{2}\sigma^2)\tau}{\sigma^2\sqrt{\tau}} + \sqrt{\tau} + \frac{\log(\frac{S}{K}) + (r-\frac{1}{2}\sigma^2)\tau}{\sigma^2\sqrt{\tau}}\right) \\
&= S\sqrt{\tau}\varphi(d_+)\,. \\
\Theta &= -\frac{\partial V_t}{\partial \tau} \\
&= -\Big[S\varphi(d_+)\left(\frac{(r+\frac{1}{2}\sigma^2)}{\sigma\sqrt{\tau}} - \frac{1}{2}\frac{\log(\frac{S}{K}) + (r+\frac{1}{2}\sigma^2)\tau}{\sigma\tau^{3/2}}\right) \\
&\qquad + rKe^{-r\tau}\mathcal{N}(d_-) - Ke^{-r\tau}\varphi(d_-)\left(\frac{(r-\frac{1}{2}\sigma^2)}{\sigma\sqrt{\tau}} - \frac{1}{2}\frac{\log(\frac{S}{K}) + (r-\frac{1}{2}\sigma^2)\tau}{\sigma\tau^{3/2}}\right)\Big] \\
&= -rKe^{-r\tau}\mathcal{N}(d_-) - \frac{S\sigma\varphi(d_+)}{2\sqrt{\tau}}\,. \\
\rho &= \frac{\partial V_t}{\partial r} \\
&= S\varphi(d_+)\frac{\tau}{\sigma\sqrt{\tau}} + \tau Ke^{-r\tau}\mathcal{N}(d_-) - Ke^{-r\tau}\varphi(d_-)\frac{\tau}{\sigma\sqrt{\tau}} \\
&= \tau Ke^{-r\tau}\mathcal{N}(d_-)\,.
\end{aligned}
$$

7. Give some remarks on Δ, Γ, Vega, and Θ?

   We remark that $0 < \Delta_C < 1$ and $-1 < \Delta_P < 0$, $\Gamma_C = \Gamma_P > 0$, the vegas are the same and both positive, etc.

   We conclude that

   - *The position in the risky asset in the replication portfolio of a European call option is between 0 and 1.*
   - *The prices of both types of options increases simultaneously with the volatility.*
   - *The function C is increasing and convex in the variable S_t and (usually) decreasing in the variable $T-t$ (for r positive).*
   - *Etc.*

Exercise 2: An numerical example of an application of the Black–Scholes model

A stock price follows geometric Brownian motion with an expected return of 16% and a volatility of 35%. The current price is \$38.
(a) What is the probability that a European call option with strike price of \$40 maturing in three months will be exercised? What is the value of this option at maturity (assume 10%p.a. risk-free interest rate)?
(b) Answer the same questions for a European put option?
(a) $S_T = S_0 e^{\eta}$ *with* $\eta \sim \mathcal{N}((\mu - \frac{1}{2}\sigma^2)T, \sigma^2 T)$. *Then* $\eta \sim \mathcal{N}((0.16 - \frac{1}{2}0.35^2)0.25, 0.35^2 0.25) \sim \mathcal{N}(0.0247, 0.0306)$ *and*

$$\begin{aligned}\mathbb{P}(S_T > 40) = \mathbb{P}(38e^{\eta} > 40) = \mathbb{P}(\eta > 0.0513) &= 1 - \mathcal{N}((0.0513 - 0.0247)/\sqrt{0.0306}) \\ &= 1 - \mathcal{N}(0.152) = 1 - 0.5604 = 0.4396\,.\end{aligned}$$

Then

$$\begin{aligned} d_+ &= \frac{\log(38/40) + (0.1 + \frac{1}{2}0.35^2)0.25}{0.35\sqrt{0.25}} = -0.06275\,, \\ d_- &= d_+ - \sigma\sqrt{T} = -0.23775\,, \end{aligned}$$

and $c = 38\mathcal{N}(-0.06275) - 40e^{-0.1\times 0.25}\mathcal{N}(-0.23775) = 2.209$.
(b) A put option will be exercised with probability

$$\mathbb{P}(S_T < 40) = 1 - \mathbb{P}(S_T \geq 40) = 1 - 0.4396 = 0.5604\,.$$

Its value is $p = 40e^{-0.1\times 0.25}\mathcal{N}(0.23775) - 38\mathcal{N}(0.06275) = 3.22$.

Exercise 3: Black–Scholes Model: Another financial derivative

Compute the price and describe the replication strategy at $t = 0$ of the European derivative being defined by the following payoff function (where we assume that the asset price process follows a geometric Brownian motion and where $k > 0$ is a positive constant).

$$f(S_T) = \max(S_T, k)\,.$$

With the usual notation, i.e.

$$d_\pm = \frac{\log\frac{S_0}{K} + \left(r \pm \frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}}\,,$$

we have

$$\begin{aligned} V_0 &= e^{-rT}\mathbb{E}_{\mathbb{Q}}(\max(S_T, k)) = e^{-rT}\mathbb{E}_{\mathbb{Q}}(S_T - k)_+ + e^{-rT}k \\ &= e^{-rT}k + S_0\mathcal{N}(d_+) - ke^{-rT}\mathcal{N}(d_-) \\ &= S_0\mathcal{N}(d_+) + e^{-rT}k\mathcal{N}(-d_-)\,. \end{aligned}$$

Hence, by a slight abuse of notation and by using Exercise 1 we have

$$H_0 = \frac{\partial V_0}{\partial S_0} = \mathcal{N}(d_+)\,,$$

and hence,

$$H_0^0 = V_0 - H_0 S_0 = e^{-rT}k(1 - \mathcal{N}(d_-))\,.$$

Exercise 4: Brownian motion

Let (W_t) be a standard Brownian motion. Which one of the following processes are standard Brownian motions as well (justify your answers)?

1. The process (X_t), being defined by $X_t = 2(W_{1+\frac{t}{4}} - W_1)$.

   We have to verify the hypothesis of a standard Brownian motion.

   (a) *Since the trajectories of* (W_t) *are a.s. continuous,* (X_t) *is a process with a.s. continuous trajectories and* $X_0 = 0$.

   (b) *Now consider the increments. We have* $X_t - X_s = 2(W_{1+\frac{t}{4}} - W_{1+\frac{s}{4}})$ *for* $t > s$. *We remark that* X_s *is* $\mathcal{F}_{1+\frac{s}{4}}$ *measurable, and that* $X_t - X_s$*, as being an increment of a Brownian motion, is independent of* $\mathcal{F}_{1+\frac{s}{4}}$.

   (c) *Furthermore, we derive that* $X_t - X_s$ *is centered (around* 0*) and normally distributed with variance* $4(1 + \frac{t}{4} - 1 - \frac{s}{4}) = t - s$.

   Hence, (X_t) *is a standard Brownian motion with respect to the filtration being defined by* $\mathcal{F}_{1+\frac{t}{4}}$.

2. The process (Y_t), being defined by $Y_t = \sqrt{t} W_1$.

   We remark that the law is indeed centered (around 0*) and the one of a normal distribution. However,* $Y_4 = 2Y_1$*, hence, the increments are not independent. Thus,* (Y_t) *is not a Brownian motion.*

3. The process (Z_t), being defined by $Z_t = W_{2t} - W_t$.

   Again we check the hypothesis of a Brownian motion.

   (a) *Clearly, in view of the continuity of the sample paths of* (W_t) *we can conclude that the sample paths of* (Z_t) *are continuous as well.*

   (b) *We consider the increments. We have* $Z_t - Z_s = W_{2t} - W_t - (W_{2s} - W_s)$ *with* $t > s$. *We remark that* Z_s *is* $\mathcal{F}_{2s}$ *measurable, but* $Z_t - Z_s$ *is not independent of* $\mathcal{F}_{2s}$. *E.g. for* $t < 2s$ *we have that* $W_{2t} - W_{2s}$ *is independent of* $\mathcal{F}_{2s}$ *while* $W_t - W_s$ *is* $\mathcal{F}_{2s}$ *measurable.*

   Hence, (Z_t) *is not a Brownian motion.*