Exercises Martingales in Financial Mathematics: Aspects of Brownian Motion and Barrier Options (Solutions)

Week 7, 2025

Exercise 1: Reflection principle

Let (W_t) be a standard Brownian motion and $(\mathcal{F}_t)$ the corresponding Brownian filtration (as introduced in the lecture course). Let M_t be the corresponding running maximum, i.e. $M_t = \sup_{s \leq t} W_s$. Derive by a heuristic argument or by a proof that for $w \leq m$, $m > 0$

$$\mathbb{P}(M_t \geq m, W_t \leq w) = \mathbb{P}(W_t \geq 2m - w)\,.$$[1]

A heuristic argument: Fix a positive level m and a positive time t. In order to "count" / "understand" the Brownian motion paths that reach level m at or before time t note that there are two types of such paths: those that reach level m prior to t but at time t are at some level w below m, and those that exceed level m at time t (there are also Brownian motion paths that are exactly at level m at time t, but the probability of these paths is vanishing, i.e. we may ignore this possibility). For each Brownian motion path that reaches level m prior to time t but is at a level w below m at time t, there is a "reflected path" that is at level $2m - w$ at time t, where this reflected path is constructed by switching the up and down moves of the Brownian motion from time $T_m = \inf\{t : W_t = m\}$ onward (it is recommendable to draw a picture). Of course, the probability that a Brownian motion path ends at exactly w or at exactly $2m - w$ is zero. Hence, in order to have nonzero probabilities consider the paths that reach level m prior to time t and are ***at or below*** *level w at time t, and consider their reflections, which are* ***at or above*** *$2m - w$ at time t. This leads to the key* reflection equality

$$\mathbb{P}(T_m \leq t, W_t \leq w) = \mathbb{P}(W_t \geq 2m - w)\,.$$

By noticing that the events $\{T_m \leq t\}$ coincides with $\{M_t \geq m\}$ for $m > 0$ we end up with the claim.

More formally: Let $S_m = \inf\{t : W_t \geq m\}$ be the first time that the Brownian motion (W_t) is greater than $m > 0$. This is an $(\mathcal{F}_t)$-stopping time and note that $\{S_m \leq t\} = \{M_t \geq m\}$ where by the continuity of Brownian motion paths

$$S_m = T_m = \inf\{t : W_t = m\}$$

[1] Hints for a proof: For a proof use the fact that for a stopping time τ with $\mathbb{P}(\tau < \infty) > 0$ we have that conditionally on $\{\tau < \infty\}$ the process $(W_{t+\tau} - W_\tau, t \geq 0)$ is a $(\mathcal{F}_{\tau+t})$-Brownian motion independent of $\mathcal{F}_\tau$ and that for positive m $S_m := \inf\{t : W_t \geq m\} = \inf\{t : W_t = m\} =: T_m$ is/are a.s. finite stopping time(s) (you are not expected to proof that, the interested student is referred e.g. to Revuz and Yor, Continuous Martingales and Brownain Motion, Sec. 3, Ch. II and Sec. 3, Ch. III; or for a rather elementary proof of the strong Markov Property to Th. 32 in P. E. Protter, Stochastic Integral and Differential Equations (Version 2.1) and for T_m to be a stopping time to Th. 4 in the same book).

and $W_{T_m} = m$. *Thus,*

$$\begin{aligned}\mathbb{P}(M_t \geq m, W_t \leq w) &= \mathbb{P}(T_m \leq t, W_t \leq w) = \mathbb{P}(T_m \leq t\,, W_t - W_{T_m} \leq w - m) \\ &= \mathbb{E}(\mathbb{1}_{W_t - W_{T_m} \leq w-m} \mathbb{1}_{T_m \leq t}) = \mathbb{E}(\mathbb{1}_{T_m \leq t} \mathbb{E}(\mathbb{1}_{W_t - W_{T_m} \leq w-m} | T_m)) = \mathbb{E}(\mathbb{1}_{T_m \leq t} \varphi(T_m))\,,\end{aligned}$$

where, by the corresponding Proposition discussed in the lecture course and by the hint,

$$\varphi(x) = \mathbb{E}(\mathbb{1}_{\tilde{W}_{t-x} \leq w-m})\,,$$

where $(\tilde{W}_s) = (W_{T_m+s} - W_{T_m}\,, s \geq 0)$ *is a Brownian motion independent of* $\mathcal{F}_{T_m}$. *Since* $(-\tilde{W}_s)$ *is also a Brownian motion we have that*

$$\varphi(x) = \mathbb{E}(\mathbb{1}_{\tilde{W}_{t-x} \leq w-m}) = \mathbb{E}(\mathbb{1}_{(-\tilde{W}_{t-x}) \leq w-m}) = \mathbb{E}(\mathbb{1}_{\tilde{W}_{t-x} \geq m-w})\,.$$

Hence,

$$\begin{aligned}\mathbb{E}(\mathbb{1}_{T_m \leq t} \varphi(T_m)) &= \mathbb{E}(\mathbb{1}_{T_m \leq t} \mathbb{E}(\mathbb{1}_{W_t - W_{T_m} \geq m-w} | T_m)) = \mathbb{P}(T_m \leq t, W_t - W_{T_m} \geq m - w) \\ &= \mathbb{P}(M_t \geq m, W_t \geq 2m - w)\,.\end{aligned}$$

Thus, by summing up we obtain

$$\mathbb{P}(M_t \geq m, W_t \leq w) = \mathbb{P}(W_t \geq 2m - w, M_t \geq m) = \mathbb{P}(W_t \geq 2m - w)\,,$$

where the last equality is obtained by noticing that $w \leq m$ *implies* $m \leq 2m - w$.

Exercise 2: Joint distribution of W_t and M_t

For a $t > 0$, find the joint probability density function of $M_t = \sup_{s \in [0,t]} W_s$ and W_t for $w \leq m$, $m > 0$.
Hint: Use Exercise 1.

By Exercise 1 we have $\mathbb{P}(M_t \geq m, W_t \leq w) = \mathbb{P}(W_t \geq 2m - w)$ *for* $m > 0$, $w \leq m$. *Furthermore, we will immediately see that a density exists, so that we also have that*

$$\begin{aligned}\mathbb{P}(M_t \geq m, W_t \leq w) &= \int_m^\infty \int_{-\infty}^w f(x,y) dy dx\,, \\ \mathbb{P}(W_t \geq 2m - w) &= \frac{1}{\sqrt{2\pi t}} \int_{2m-w}^\infty e^{\frac{-z^2}{2t}}\, dz = 1 - \frac{1}{\sqrt{2\pi t}} \int_{-\infty}^{2m-w} e^{\frac{-z^2}{2t}}\, dz\,.\end{aligned}$$

Hence,

$$\int_m^\infty \int_{-\infty}^w f(x,y) dy dx = 1 - \frac{1}{\sqrt{2\pi t}} \int_{-\infty}^{2m-w} e^{\frac{-z^2}{2t}}\, dz\,,$$

being differentiable in m. *Differentiating with respect to* m *yields*

$$-\int_{-\infty}^w f(m,y) dy = -\frac{2}{\sqrt{2\pi t}} e^{-\frac{(2m-w)^2}{2t}}\,,$$

being differentiable in w. *Differentiating with respect to* w *then yields, up to a sign, the joint density for* $m > 0$, $w \leq m$, *i.e.*

$$-f(m,w) = -\frac{2(2m-w)}{t\sqrt{2\pi t}} e^{-\frac{(2m-w)^2}{2t}}\,.$$

Remark: Observe that from the reflection principle it also follows that for $m > 0$, $w \leq m$

$$\begin{aligned}\mathbb{P}(M_t \leq m\,, W_t \leq w) &= \mathbb{P}(W_t \leq w) - \mathbb{P}(M_t \geq m\,, W_t \leq w)\\ &= \mathbb{P}(W_t \leq w) - \mathbb{P}(W_t \geq 2m - w) = \mathcal{N}\left(\frac{w}{\sqrt{t}}\right) - \mathcal{N}\left(\frac{w-2m}{\sqrt{t}}\right). \end{aligned} \tag{1}$$

Furthermore, note that for $0 < m \leq w$ we get by using $M_t \geq W_t$

$$\mathbb{P}(M_t \leq m\,, W_t \leq w) = \mathbb{P}(M_t \leq m\,, W_t \leq m) = \mathbb{P}(M_t \leq m)\,,$$

and by substituting $w = m$ in (1) we end up (for $0 < m \leq w$) with

$$\mathbb{P}(M_t \leq m\,, W_t \leq w) = \mathbb{P}(M_t \leq m) = \mathcal{N}\left(\frac{m}{\sqrt{t}}\right) - \mathcal{N}\left(\frac{-m}{\sqrt{t}}\right), \tag{2}$$

while for $m \leq 0$ we have $M_t \geq M_0 = 0$. We finally conclude that for $m \leq 0$ we have $\mathbb{P}(M_t \leq m\,, W_t \leq w) = 0$.

Remark (side notes): Note that (2) also implies for a fixed t the well known fact that

$$\mathbb{P}(M_t \leq m) = \mathcal{N}\left(\frac{m}{\sqrt{t}}\right) - \mathcal{N}\left(\frac{-m}{\sqrt{t}}\right) = \mathbb{P}(W_t \leq m) - \mathbb{P}(W_t \leq -m) = \mathbb{P}(|W_t| \leq m)\,.$$

Furthermore, by changing variables the joint density of M_t and $Y_t = M_t - W_t$ can be easily derived. Since this density shows that (M_t, Y_t) is exchangeable it follows that $M_t \sim Y_t$.

Exercise 3: Brownian motion with drift

Let $(W_t)_{t\in[0,T]}$ be a standard Brownian motion defined on $(\Omega\,, \mathcal{F}, \mathbb{Q})$ and let $\hat{W} = (\alpha t + W_t)_{t\in[0,T]}$ for a given real α, i.e. the Brownian motion $(\hat{W}_t)$ has drift α under $\mathbb{Q}$. We further define $\hat{M}_T = \sup_{0\leq t\leq T} \hat{W}_t$. Show that for $m > 0$, $w \leq m$, the joint density function of $(\hat{M}_T, \hat{W}_T)$ under $\mathbb{Q}$ is given by

$$\tilde{f}(m,w) = \frac{2(2m-w)}{T\sqrt{2\pi T}} e^{\alpha w - \frac{1}{2}\alpha^2 T - \frac{1}{2T}(2m-w)^2}\,. \tag{3}$$

Hint: Use Girsanov for the density $\hat{Z}_t = e^{-\alpha W_t - \frac{1}{2}\alpha^2 t}$ and the result from Exercise 2.

Following the hint define the exponential martingale

$$\hat{Z}_t = e^{-\alpha W_t - \frac{1}{2}\alpha^2 t} = e^{-\alpha \hat{W}_t + \frac{1}{2}\alpha^2 t}\,, \quad 0 \leq t \leq T\,,$$

and use $\hat{Z}_T$ in order to define a new probability measure $\mathbb{P}$ by

$$\mathbb{P}(A) = \int_A \hat{Z}_T d\mathbb{Q} \quad \textit{for all } A \in \mathcal{F}\,,$$

i.e. $\frac{d\mathbb{P}}{d\mathbb{Q}} = \hat{Z}_T$ (being sampled from a "true" martingale). From Girsanov's Theorem we know that $(\hat{W}_t)$ is a Brownian motion (with zero drift) under $\mathbb{P}$. Hence, Exercise 2 yields the joint density of $(\hat{M}_T, \hat{W}_T)$ under $\mathbb{P}$, which is

$$\hat{f}(m,w) = \frac{2(2m-w)}{T\sqrt{2\pi T}} e^{-\frac{(2m-w)^2}{2T}}$$

for $w \leq m$, $m > 0$ (and vanishing otherwise). Hence, by changing measure

$$\begin{aligned}\mathbb{Q}(\hat{M}_T \leq m\,, \hat{W}_T \leq w) &= \mathbb{E}_{\mathbb{Q}}(1\!\!1_{\hat{M}_T \leq m\,, \hat{W}_T \leq w}) \\ &= \mathbb{E}_{\mathbb{P}}\left[\frac{1}{\hat{Z}_T} 1\!\!1_{\hat{M}_T \leq m\,, \hat{W}_T \leq w}\right] \\ &= \mathbb{E}_{\mathbb{P}}\left[e^{\alpha \hat{W}_T - \frac{1}{2}\alpha^2 T} 1\!\!1_{\hat{M}_T \leq m\,, \hat{W}_T \leq w}\right] \\ &= \int_{-\infty}^{w} \int_0^m e^{\alpha y - \frac{1}{2}\alpha^2 T} \hat{f}(x,y) dx dy\,.\end{aligned}$$

Therefor for $m > 0$ and $w \leq m$ the density of $(\hat{M}_T\,, \hat{W}_T)$ under $\mathbb{Q}$ is

$$\tilde{f}(m,w) = e^{\alpha w - \frac{1}{2}\alpha^2 T} \hat{f}(m,w)\,,$$

i.e. (3) (while otherwise the density is vanishing).

Exercise 4: Value of a up-and-out call

With the usual notation, price (at $t = 0$) the following so-called up-and-out call being defined by

$$h = (S_T - k)_+ 1\!\!1_{S_t < b\,, \forall t \in [0,T]}$$

where we assume that $S_0 < b$ and $0 < k < b$ (otherwise, the option must be knocked out in order to be in the money and hence, could only pay off zero).

Since h is a square integrable $\mathcal{F}_T$-measurable random variable the price V_0 at $t = 0$ is given by

$$V_0 = e^{-rT} \mathbb{E}_{\mathbb{Q}}((S_T - k)_+ 1\!\!1_{S_t < b\,, \forall t \in [0,T]})\,,$$

with

$$S_t = S_0 \exp((r - \frac{1}{2}\sigma^2)t + \sigma W_t) = S_0 e^{\sigma \hat{W}_t}\,,$$

where $\hat{W}_t = \alpha t + W_t$,

$$\alpha = \frac{1}{\sigma}(r - \frac{1}{2}\sigma^2)\,.$$

Hence, with Exercise 3 and with the notation

$$\tilde{b} = \frac{1}{\sigma} \log\big(\frac{b}{S_0}\big)\,, \quad \tilde{k} = \frac{1}{\sigma} \log\big(\frac{k}{S_0}\big)\,,$$

$$\begin{aligned}V_0 &= e^{-rT} \mathbb{E}_{\mathbb{Q}}[(S_0 e^{\sigma \hat{W}_T} - k) 1\!\!1_{k < S_0 e^{\sigma \hat{W}_T} < b} 1\!\!1_{S_0 e^{\sigma \hat{W}_t} < b,\ \forall t \in [0,T]}] \\ &= e^{-rT} \int_{\tilde{k}}^{\tilde{b}} \int_{w+}^{\tilde{b}} (S_0 e^{\sigma w} - k) \frac{2(2m - w)}{T\sqrt{2\pi T}} \exp\big(\alpha w - \frac{1}{2}\alpha^2 T - \frac{1}{2T}(2m - w)^2\big) dm dw \\ &= e^{-rT} \int_{\tilde{k}}^{\tilde{b}} (S_0 e^{\sigma w} - k) \frac{e^{\alpha w - \frac{1}{2}\alpha^2 T}}{\sqrt{2\pi T}} \int_{w+}^{\tilde{b}} \frac{2(2m - w)}{T} e^{-\frac{1}{2T}(2m - w)^2} dm dw \\ &= e^{-rT} \int_{\tilde{k}}^{\tilde{b}} (S_0 e^{\sigma w} - k) \frac{e^{\alpha w - \frac{1}{2}\alpha^2 T}}{\sqrt{2\pi T}} (-e^{-\frac{1}{2T}(2m-w)^2})\big|_{w+}^{\tilde{b}} dw \\ &= \int_{\tilde{k}}^{\tilde{b}} (S_0 e^{\sigma w} - k) \frac{e^{-rT + \alpha w - \frac{1}{2}\alpha^2 T - \frac{1}{2T} w^2}}{\sqrt{2\pi T}} dw - \int_{\tilde{k}}^{\tilde{b}} (S_0 e^{\sigma w} - k) \frac{e^{-rT + \alpha w - \frac{1}{2}\alpha^2 T - \frac{1}{2T}(2\tilde{b} - w)^2}}{\sqrt{2\pi T}} dw \\ &= S_0 I_1 - k I_2 - S_0 I_3 + k I_4\,,\end{aligned}$$

where

$$I_1 = \frac{1}{\sqrt{2\pi T}} \int_{\tilde{k}}^{\tilde{b}} e^{\sigma w - rT + \alpha w - \frac{1}{2}\alpha^2 T - \frac{1}{2T} w^2}\, dw\,,$$

$$I_2 = \frac{1}{\sqrt{2\pi T}} \int_{\tilde{k}}^{\tilde{b}} e^{-rT + \alpha w - \frac{1}{2}\alpha^2 T - \frac{1}{2T} w^2}\, dw\,,$$

$$\begin{aligned} I_3 &= \frac{1}{\sqrt{2\pi T}} \int_{\tilde{k}}^{\tilde{b}} e^{\sigma w - rT + \alpha w - \frac{1}{2}\alpha^2 T - \frac{1}{2T}(2\tilde{b} - w)^2}\, dw \\ &= \frac{1}{\sqrt{2\pi T}} \int_{\tilde{k}}^{\tilde{b}} e^{\sigma w - rT + \alpha w - \frac{1}{2}\alpha^2 T - \frac{2\tilde{b}^2}{T} + \frac{2\tilde{b}w}{T} - \frac{w^2}{2T}}\, dw \end{aligned}$$

$$I_4 = \frac{1}{\sqrt{2\pi T}} \int_{\tilde{k}}^{\tilde{b}} e^{-rT + \alpha w - \frac{1}{2}\alpha^2 T - \frac{2\tilde{b}^2}{T} + \frac{2\tilde{b}w}{T} - \frac{w^2}{2T}}\, dw\,.$$

Observe that each of these integrals are of the form

$$\frac{1}{\sqrt{2\pi T}} \int_{\tilde{k}}^{\tilde{b}} e^{\beta + \gamma w - \frac{w^2}{2T}}\, dw = \frac{1}{\sqrt{2\pi T}} \int_{\tilde{k}}^{\tilde{b}} e^{-\frac{(w-\gamma T)^2}{2T} + \frac{\gamma^2 T}{2} + \beta} dw\,.$$

By substituting $y = \frac{w - \gamma T}{\sqrt{T}}$ *we obtain*

$$\begin{aligned} \frac{1}{\sqrt{2\pi T}} \int_{\tilde{k}}^{\tilde{b}} e^{\beta + \gamma w - \frac{w^2}{2T}}\, dw &= \frac{e^{\frac{1}{2}\gamma^2 T + \beta}}{\sqrt{2\pi}} \int_{\frac{\tilde{k} - \gamma T}{\sqrt{T}}}^{\frac{\tilde{b} - \gamma T}{\sqrt{T}}} e^{-\frac{1}{2}y^2}\, dy \\ &= e^{\frac{1}{2}\gamma^2 T + \beta} \left(\mathcal{N}\left(\frac{\tilde{b} - \gamma T}{\sqrt{T}} \right) - \mathcal{N}\left(\frac{\tilde{k} - \gamma T}{\sqrt{T}} \right) \right) \\ &= e^{\frac{1}{2}\gamma^2 T + \beta} \left(1 - \mathcal{N}\left(\frac{-\tilde{b} + \gamma T}{\sqrt{T}} \right) - \left(1 - \mathcal{N}\left(\frac{-\tilde{k} + \gamma T}{\sqrt{T}} \right) \right) \right) \\ &= e^{\frac{1}{2}\gamma^2 T + \beta} \left(\mathcal{N}\left(\frac{\log\left(\frac{S_0}{k}\right) + \gamma\sigma T}{\sigma\sqrt{T}} \right) - \mathcal{N}\left(\frac{\log\left(\frac{S_0}{b}\right) + \gamma\sigma T}{\sigma\sqrt{T}} \right) \right)\,. \end{aligned}$$

It is efficient to introduce the following notation

$$\delta_{\pm}(s) = \frac{\log(s) + (r \pm \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}\,.$$

For I_1*, we have that* $\beta = -rT - \frac{1}{2}\alpha^2 T$ *and* $\gamma = \alpha + \sigma$ *so that* $\frac{1}{2}\gamma^2 T + \beta = 0$ *and* $\gamma\sigma = r + \frac{1}{2}\sigma^2$*. Hence,*

$$I_1 = \mathcal{N}\left(\delta_+\left(\frac{S_0}{k} \right) \right) - \mathcal{N}\left(\delta_+\left(\frac{S_0}{b} \right) \right)\,.$$

For I_2*, we have* $\beta = -rT - \frac{1}{2}\alpha^2 T$ *and* $\gamma = \alpha$*. Hence,* $\frac{1}{2}\gamma^2 T + \beta = -rT$ *and* $\gamma\sigma = r - \frac{1}{2}\sigma^2$*. Therefore,*

$$I_2 = e^{-rT} \left[\mathcal{N}\left(\delta_-\left(\frac{S_0}{k} \right) \right) - \mathcal{N}\left(\delta_-\left(\frac{S_0}{b} \right) \right) \right]\,.$$

For I_3*, we obtain* $\beta = -rT - \frac{1}{2}\alpha^2 T - \frac{2\tilde{b}^2}{T}$ *and* $\gamma = \alpha + \sigma + \frac{2\tilde{b}}{T}$*, so that*

$$\begin{aligned} \frac{1}{2}\gamma^2 T + \beta &= (-\frac{2r}{\sigma^2} - 1) \log\left(\frac{S_0}{b}\right) \\ \gamma\sigma T &= (r + \frac{1}{2}\sigma^2)T + 2\log\left(\frac{b}{S_0}\right)\,. \end{aligned}$$

In view of that we have

$$I_3 = \left(\frac{S_0}{b}\right)^{-\frac{2r}{\sigma^2}-1}\left[\mathcal{N}\left(\delta_+\left(\frac{b^2}{kS_0}\right)\right) - \mathcal{N}\left(\delta_+\left(\frac{b}{S_0}\right)\right)\right].$$

It remains I_4 *where we have* $\beta = -rT - \frac{1}{2}\alpha^2 T - \frac{2\tilde{b}^2}{T}$ *and* $\gamma = \alpha + \frac{2\tilde{b}}{T}$, *so that*

$$\begin{aligned}\frac{1}{2}\gamma^2 T + \beta &= -rT + (-\frac{2r}{\sigma^2}+1)\log\big(\frac{S_0}{b}\big)\\ \gamma\sigma T &= (r-\frac{1}{2}\sigma^2)T + 2\log\big(\frac{b}{S_0}\big).\end{aligned}$$

Hence,

$$I_4 = e^{-rT}\left(\frac{S_0}{b}\right)^{-\frac{2r}{\sigma^2}+1}\left[\mathcal{N}\left(\delta_-\left(\frac{b^2}{kS_0}\right)\right) - \mathcal{N}\left(\delta_-\left(\frac{b}{S_0}\right)\right)\right].$$

By summing up we end up with the price of a up-and-out call (under the stated parameter restrictions), i.e.

$$\begin{aligned}V_0 =& S_0\left[\mathcal{N}\left(\delta_+\left(\frac{S_0}{k}\right)\right) - \mathcal{N}\left(\delta_+\left(\frac{S_0}{b}\right)\right)\right]\\ &- e^{-rT}k\left[\mathcal{N}\left(\delta_-\left(\frac{S_0}{k}\right)\right) - \mathcal{N}\left(\delta_-\left(\frac{S_0}{b}\right)\right)\right]\\ &- b\left(\frac{S_0}{b}\right)^{-\frac{2r}{\sigma^2}}\left[\mathcal{N}\left(\delta_+\left(\frac{b^2}{kS_0}\right)\right) - \mathcal{N}\left(\delta_+\left(\frac{b}{S_0}\right)\right)\right]\\ &+ e^{-rT}k\left(\frac{S_0}{b}\right)^{-\frac{2r}{\sigma^2}+1}\left[\mathcal{N}\left(\delta_-\left(\frac{b^2}{kS_0}\right)\right) - \mathcal{N}\left(\delta_-\left(\frac{b}{S_0}\right)\right)\right].\end{aligned}$$