Exercises Martingales in Financial Mathematics: Model CRR and towards Black Scholes (Solutions)

Week 5, 2025

Exercise 1: Model CRR

The Cox Ross Rubinstein (binomial) model is sometimes used in order to price American options. Recall that there is only one risky asset with price S_n at n for all $n \in \{0, 1, \ldots, N\}$ along with a risk-less asset. The risk-free interest rate for each time period is given by r. The price process (S_n) can be modeled by the relative variations of the quotes over the time subperiods, being denoted by a and b with $-1 < a < b$, i.e.

$$S_{n+1} = \begin{cases} S_n\,(1+a) \\ S_n\,(1+b) \end{cases} .$$

For our example we fix $N = 3$, $r = 0.02$, $b = 0.1$, $a = -0.1$, and $S_0 = 100$.

1. Is this particular market viable? Is it complete?

   Yes, this particular market is viable since $-0.1. < 0.02 < 0.1$, *i.e.* $a < r < b$, *see Exercise 1, Week 2. We have also seen that in such a viable case, the corresponding market is also complete, see again Exercise 1, Week 2.*

2. Describe the martingale measure $\mathbb{Q}$.

   A short calculation shows $p^* = \mathbb{Q}[S_{n+1}/S_n = 1 + a|\mathcal{F}_n] = 0.4$.

3. Draw a graph of the tree representing the possible values of S_n.

   Cf. the attached file.

4. We intend to derive the price of an American put, where the maturity is represented by N and where the strike is given by $K = 95$. Draw a graph of the tree with the possible payoffs at n, i.e. with the used notation visualize Z_n for each possible value of S_n.

   Cf. the attached file.

5. Starting at $n = N$, construct the Snell envelope of Z_n.

   Cf. the attached file.

Exercise 2: The SDE of a GBM

We are interested in the following equation

$$S_t = x_0 + \mu \int_0^t S_s\,ds + \sigma \int_0^t S_s\,dW_s\,, \quad t \in [0, T]. \tag{1}$$

1. Assume
$$S_t = x_0\, e^{\left(\mu - \frac{\sigma^2}{2}\right) t + \sigma W_t}\,.$$
By using the Itô formula, show that (S_t) is a solution of (1).

Consider S_t as a function in t and W_t
$$S(t,\, x) = x_0\, e^{\left(\mu - \frac{\sigma^2}{2}\right) t + \sigma x}\,.$$
The first two derivatives with respect to x and the first one with respect to t are given by
$$\begin{aligned} \frac{\partial S}{\partial x}(t,x) &= \sigma\, S(t,x)\,, \\ \frac{\partial^2 S}{\partial x^2}(t,x) &= \sigma^2\, S(t,x)\,, \\ \frac{\partial S}{\partial t}(t,x) &= \left(\mu - \frac{\sigma^2}{2}\right)\, S(t,x)\,. \end{aligned}$$
By applying the Itô formula we end up with
$$S_t = x_0 + \int_0^t \left(\mu - \frac{\sigma^2}{2}\right) S_s\, ds + \int_0^t \sigma\, S_s\, dW_s + \frac{1}{2}\int_0^t \sigma^2\, S_s\, ds\,,$$
i.e. with a solution for the SDE (1).

2. Show that (Z_t) defined by $Z_t = x_0/S_t$ solves
$$Z_t = 1 + \mu' \int_0^t Z_s ds + \sigma' \int_0^t Z_s dW_s,$$
where μ' and σ' are constants, which have to be identified.

As before
$$Z_t = 1 - \int_0^t \left(\mu - \frac{\sigma^2}{2}\right) Z_s\, ds - \int_0^t \sigma\, Z_s\, dW_s + \frac{1}{2}\int_0^t \sigma^2\, Z_s\, ds\,.$$
Hence, $\sigma' = -\sigma$ and $\mu' = -\mu + \sigma^2$.

3. Assume that (Y_t) is another solution of the SDE (1). Show that $d(Y_t Z_t) = 0$. Derive the uniqueness of the solution of (1). In finance, the unique solution of this equation is called Black and Scholes model.

The integration by parts formula yields
$$d(Y_t\, Z_t) = Z_t\, dY_t + Y_t\, dZ_t + d\langle Y,\, Z\rangle_t\,.$$
With a little calculation we now end up with $d(Y_t\, Z_t) = 0$. We conclude $Y_t\, Z_t = Y_0 Z_0$. Hence, (S_t) is the unique solution of the SDE.