Exercises Martingales in Financial Mathematics: Further aspects of discrete time models (Solutions)

Week 4, 2025

Exercise 1: An example for an incomplete model

Consider a one-period model with one risky asset. The spot price of this risky asset is 10. Assume that after the period, i.e. at $n = 1$, $S_1 \in \{9, 10, 12\}$ holds. Furthermore, assume that the risk-free interest rate is vanishing. Finally, let $H(S_1)$ be a European claim written on the risky asset S.

1. Is this market viable?

   In order to show that this market is viable it is sufficient to find a risk-neutral probability measure. If we define the vector containing the probabilities of S_1 by $\{\beta, \alpha, \gamma\} = \{1/3, 1/2, 1/6\}$ we end up with $\mathbb{E}^*[S_1] = 10$. Hence, there is a risk-neutral probability measure so that this market is viable.

2. Describe the risk-neutral probabilities.

   Again we denote the vector containing the probabilities of S_1 by $\{\beta, \alpha, \gamma\}$ and we assume $0 < \alpha, \beta, \gamma < 1$ and $\alpha + \beta + \gamma = 1$. Then $\mathbb{E}^{(\alpha,\beta,\gamma)}[S_1] = 9\,\beta + 10\,\alpha + 12\,\gamma$. The conditions for the risk-neutral probability measures are now given by $\gamma = \frac{1-\alpha}{3}$ and $\beta = 2\frac{1-\alpha}{3}$, with $0 < \alpha < 1$. From now on we work under a generic risk-neutral probability measure $\mathbb{P}^{(\alpha)}$.

3. Describe a potentially existing replicating strategy for the claim being defined by H and conclude that the strategy exists if and only if $H(12) - 3H(10) + 2H(9) = 0$.

   We want to describe the potentially existing strategy. Hence, we have $\phi^0\, S_1^0 + \phi\, S_1 = H(S_1)$. In view of that we can derive

$$\begin{cases} \phi^0 + 9\,\phi &= H(9) \\ \phi^0 + 10\,\phi &= H(10) \\ \phi^0 + 12\,\phi &= H(12) \end{cases} .$$

   This system has a solution if and only if $H(12) - 3H(10) + 2H(9) = 0$.

4. Show that the value of an option admitting a replication strategy does not depend on the choice of the risk-neutral probability measure.

   It is sufficient to show that $\mathbb{E}^{(\alpha)}[H] = \mathbb{E}^{(\alpha')}[H]$ for $\alpha \neq \alpha'$. Thus, we have

$$\begin{aligned} &2\frac{1-\alpha}{3}H(9) + \alpha H(10) + \frac{1-\alpha}{3}H(12) \\ &\quad = 2\frac{1-\alpha}{3}H(9) + \alpha H(10) + \frac{1-\alpha}{3}H(12) - \frac{\alpha'-\alpha}{3}[H(12) - 3H(10) + 2H(9)] \\ &\quad = 2\frac{1-\alpha'}{3}H(9) + \alpha' H(10) + \frac{1-\alpha'}{3}H(12)\,. \end{aligned}$$

5. Can the claims satisfying $H(12) - 3H(10) + 2H(9) = 0$ be described?

   Yes, the claims are affine linear functions of S_1; hence, economically speaking, they are given by multiples of a forward (Week 1, Exercise 1) and an investments in the risk-free asset and/or in the risky asset.

Exercise 2: Doob decomposition

Consider our finite probability space setting. Show that supermartingale (X_n) has the following decomposition

$$X_n = X_0 + M_n - A_n$$

where (M_n) is a martingale null at 0 and (A_n) is a non-decreasing predictable process, null at 0. Furthermore, show that this decomposition is unique.

We start with a sequence (X_n) (where the X_n are automatically integrable in our finite probability space setting). Then

$$\begin{aligned} X_n &= X_0 + \sum_{k=1}^{n} \Delta X_k \\ &= X_0 + \underbrace{\sum_{k=1}^{n} \mathbb{E}(\Delta X_k | \mathcal{F}_{k-1})}_{(\hat{A}_n) \text{ predictable with } \hat{A}_0 = 0} + \underbrace{\sum_{k=1}^{n} (\Delta X_k - \mathbb{E}(\Delta X_k | \mathcal{F}_{k-1}))}_{(M_n) \text{ martingale with } M_0 = 0} . \end{aligned}$$

If $X_n = X_0 + M'_n + \hat{A}'_n$ holds for another decomposition of "the same type" we have that

$$\begin{aligned} X_n - X_0 &= \hat{A}'_n + M'_n &= \hat{A}_n + M_n , \\ X_{n+1} - X_0 &= \hat{A}'_{n+1} + M'_{n+1} &= \hat{A}_{n+1} + M_{n+1} . \end{aligned}$$

Hence,

$$\hat{A}'_{n+1} - \hat{A}'_n = M_{n+1} - M_n - (M'_{n+1} - M'_n) + (\hat{A}_{n+1} - \hat{A}_n) .$$

Thus,

$$\mathbb{E}(\hat{A}'_{n+1} - \hat{A}'_n | \mathcal{F}_n) = \mathbb{E}(M_{n+1} - M_n | \mathcal{F}_n) - \mathbb{E}(M'_{n+1} - M'_n | \mathcal{F}_n) + \mathbb{E}(\hat{A}_{n+1} - \hat{A}_n | \mathcal{F}_n) ,$$

so that $\hat{A}'_{n+1} - \hat{A}'_n = \hat{A}_{n+1} - \hat{A}_n$. Since $\hat{A}'_0 = \hat{A}_0 = 0$ it follows consecutively that $\hat{A}'_n = \hat{A}_n$ for all n. Furthermore, with $M'_0 = M_0 = 0$ we conclude consecutively that also $M_n = M'_n$ for all n.

Hence,

$$X_n = X_0 + \hat{A}_n + M_n ,$$

with (M_n) being a martingale and $(\hat{A}_n)$ being a predictable process, where the decomposition is unique. For a supermartingale (X_n) we use that

$$X_{n+1} - X_n = M_{n+1} - M_n + (\hat{A}_{n+1} - \hat{A}_n) ,$$

the predictability property of $(\hat{A}_n)$, and the martingale property of (M_n) in order to conclude that

$$\mathbb{E}(X_{n+1} | \mathcal{F}_n) - X_n = \hat{A}_{n+1} - \hat{A}_n \leq 0 .$$

The left hand side of the above equation is non-positive by the supermartingale property of (X_n) and so is the right hand side. Thus, the sequence $(\hat{A}_n)$ is non-increasing so that $(A_n) = (-\hat{A}_n)$ is non-decreasing (clearly unique in this decomposition).

Exercise 3: American call

Let C_n^A, C_n be the price of an American, European option, respectively, at n with payoff being determined by $\{Z_n\}_{0\le n\le N}$, Z_N, respectively.

- Show that for all n $C_n^A \ge C_n$.

  The result can be proved by backward induction. Clearly, for the not exercised American claim we have $C_N^A = C_N$ since the payoffs of the American and the European option coincide. Now, at each n we consider the Snell envelope of the sequence $\{Z_n\}_{0\le n\le N}$. Hence, we take the maximum of Z_n and $(1+r)^{-1}\,\mathbb{E}^{\mathbb{Q}}[C_{n+1}^A \,|\, \mathcal{F}_n]$. Hence, given $C_{n+1}^A \ge C_{n+1}$, we have

$$C_n^A = \max\left\{Z_n,\, (1+r)^{-1}\,\mathbb{E}^{\mathbb{Q}}[C_{n+1}^A \,|\, \mathcal{F}_n]\right\}$$
$$\ge (1+r)^{-1}\,\mathbb{E}^{\mathbb{Q}}[C_{n+1}^A \,|\, \mathcal{F}_n] \ge (1+r)^{-1}\,\mathbb{E}^{\mathbb{Q}}[C_{n+1} \,|\, \mathcal{F}_n] = C_n\,.$$

- Show that $Z_n \le C_n$ for all n implies $C_n^A = C_n$ for all n.

  Assume $C_n \ge Z_n$, for all n. Hence, the sequence $(\tilde{C}_n)$, which is a martingale under $\mathbb{Q}$, appears to be a supermartingale under $\mathbb{Q}$ being an upper bound for the sequence $(\tilde{Z}_n)$ so that

$$\tilde{C}_n^A \le \widetilde{C}_n \quad \forall n \in \{0,1,\ldots,N\}\,.$$

- (Application: American Call) Assume that the risk-free interest rate is positive. Show that the price of an American call is equal to the price of an analogously specified European Call. *Note that*

$$\begin{aligned}
C_n(K,\,S_n) &= (1+r)^{-(N-n)}\mathbb{E}^{\mathbb{Q}}[(S_N-K)^+ \,|\, \mathcal{F}_n] \\
&= (1+r)^n\mathbb{E}^{\mathbb{Q}}[(\widetilde{S}_N - K\,(1+r)^{-N})^+ \,|\, \mathcal{F}_n] \\
&\ge (1+r)^n\mathbb{E}^{\mathbb{Q}}[\widetilde{S}_N - K\,(1+r)^{-N} \,|\, \mathcal{F}_n] \\
&= (1+r)^n\widetilde{S}_n - K(1+r)^{-(N-n)} \\
&= S_n - K\,(1+r)^{-(N-n)} \ge S_n - K\,,
\end{aligned}$$

  along with $C_n(K,S_n) \ge 0$. The claim follows by the above points. Note that up to the last step, where we have used that r is positive, the arguments remain valid in the case of puts.