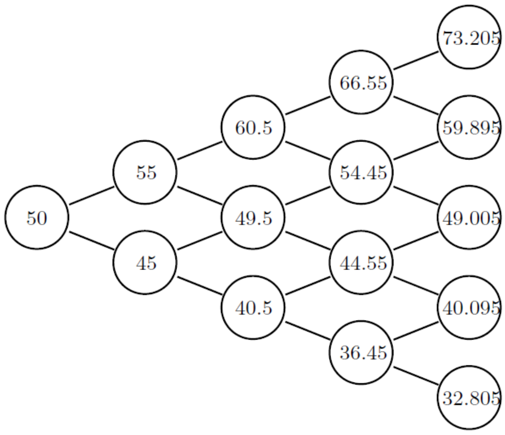
Exercises Martingales in Financial Mathematics: The complete CRR in action / on incomplete markets (Solutions)

Week 3, 2025

Exercise 1: Example

Consider the following binomial-tree of possible realizations for a price process, with time tick 1 month and interest rate $r_c = 0.2$ (continuous compounding), i.e. $N = 4$, $T = N\Delta t = 1/3$, $S^0_n = e^{r_c n \Delta t}$:



(a) Calculate the value of a call with strike \$50 at time $t = 0$.

(b) Describe the replication strategy for the following scenarios:

(i) 1. move up, 2. move down, 3. move up, 4. move up;

(ii) 1. move down, 2. move down, 3. mote up, 4. move down.

Hint.: Use the following table:

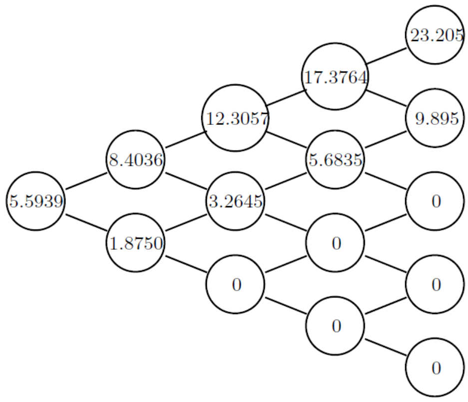
n	move	S_n	V_n	ϕ_n	ϕ^0_n	$\tilde{V}_n$	$\tilde{S}_n$
0	-	50	?	-	-	?	50

We have $u = (1 + b) = 1.1$, $d = (1 + a) = 0.9$, $\Delta t = \frac{1}{12}$, $T = \frac{4}{12}$, $p^* = \frac{u - e^{r_c \Delta t}}{u-d} = 0.41596835$, $c(n, x) = e^{-r_c \Delta t}((1 - p^*)c(n + 1, xu) + p^* c(n + 1, xd))$ *(furthermore, recall that* $c(n, S_n)$ *coincides with* $V_n = e^{-r_c(T-n\Delta t)}\mathbb{E}_{\mathbb{Q}}((S_N - K)_+|\mathcal{F}_n)$*, where* $K = 50$ *Dollars). The value of the call option can be calculated directly by*

$$c(0, S_0) = e^{-0.2\,1/3} \sum_{j=0}^{4} \binom{4}{j} \left(\frac{1.1 - e^{0.2/12}}{1.1 - 0.9}\right)^j \left(1 - \frac{1.1 - e^{0.2/12}}{1.1 - 0.9}\right)^{4-j} (50\, 0.9^j\, 1.1^{4-j} - 50)_+ = 5.5939\,.$$

Recursively we obtain the following tree of values of the call option $c(n, x)$*:*

For the replication strategy we use $\phi_{n+1} = \frac{c(n+1,S_n u)-c(n+1,S_n d)}{S_n(u-d)}$, $\phi^0_{n+1} = \tilde{V}_n - \phi_{n+1} S_n e^{-r_c n\Delta t}$, *with*



$\tilde{V}_n = V_0 + \sum_{k=1}^{n} \phi_k \Delta \tilde{S}_k$, $\tilde{S}_n = S_n e^{-r_c n\Delta t}$, *so that we can calculate for (i)*

temps n	move	S_n	V_n	ϕ_n	ϕ^0_n	$\tilde{V}_n$	$\tilde{S}_n$
0	-	50.000	5.5939	-	-	5.5939	50.0000
1	up	55.000	8.4036	0.65286	-27.0491	8.2647	54.0909
2	down	49.500	3.2645	0.82193	-36.1943	3.1575	47.8772
3	up	54.450	5.6835	0.57409	-24.3283	5.4063	51.7944
4	up	59.895	9.8950	0.90863	-41.6556	9.2568	56.0322

Note that for $n = 1, 2, 3$, $\phi_n S_n + \phi^0_n e^{r_c n\Delta t} = \phi_{n+1} S_n + \phi^0_{n+1} e^{r_c n\Delta t}$ *and that* $\phi_4 S_4 + \phi^0_4 e^{r_c 4\Delta t} =$ 9.895, *i.e. the pay-off of the call. For (ii) we have*

temps n	move	S_n	V_n	ϕ_n	ϕ^0_n	$\tilde{V}_n$	$\tilde{S}_n$
0	-	50.000	5.5939	-	-	5.5939	50.0000
1	down	45.000	1.8750	0.65286	-27.0491	1.8440	44.2562
2	down	40.500	0.0000	0.36272	-14.2086	0.0000	39.1723
3	up	44.550	0.0000	0.00000	0.0000	0.0000	42.3773
4	down	40.095	0.0000	0.00000	0.0000	0.0000	37.5092

Observe that after $n = 2$ *the development of the share is irrelevant for the replication portfolio. This is due to the fact that the call will be worthless at maturity* $\mathbb{Q}$ *a.s. and (since the measures are equivalent)* $\mathbb{P}$ *a.s.*

Exercise 2: Trinomial model

The trinomial model can be considered as being an extension of the Cox Ross Rubinstein model (binomial model). There is again only one risky asset with price S_n at n until N along with a risk-less asset with risk-free interest rate r for every time period, i.e. $S_n^0 = (1+r)^n$. However, between two consecutive periods the price changes here by a factor $1+d$ or $1+m$ or $1+u$, i.e.

$$S_{n+1} = \begin{cases} S_n\,(1+d) \\ S_n\,(1+m) \\ S_n\,(1+u) \end{cases},$$

where $-1 < d < m < u$. Suppose that the initial stock price is given by S_0 and define the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with $\Omega = \{1+d, 1+m, 1+u\}^N$, $\mathcal{F} = \mathcal{P}(\Omega)$ and $\mathbb{P}$ being a probability measure such that $\mathbb{P}(\omega) > 0$ for every atom ω. Furthermore, again for $n = 1, \ldots, N$ the σ-algebra $\mathcal{F}_n$ is generated by the random variables $S_1, \ldots, S_n$, i.e. $\mathcal{F}_n = \sigma(S_1, \ldots, S_n)$ ($\mathcal{F}_0 = \{\Omega, \varnothing\}$). Finally, we define $T_n = S_n/S_{n-1}$, with possible values $1+d$, $1+m$ and $1+u$, and we assume that the T_i are i.i.d.

1. Show that in order to end up with a viable market it is necessary that $r \in]d, u[$.

   We define a new family of probability measures $\mathbb{Q}^{(\alpha,\beta,\gamma)}$ by $\mathbb{Q}^{(\alpha,\beta,\gamma)}[T_i = 1+d] = \alpha$, $\mathbb{Q}^{(\alpha,\beta,\gamma)}[T_i = 1+m] = \beta$ and $\mathbb{Q}^{(\alpha,\beta,\gamma)}[T_i = 1+u] = \gamma$. It is easy to see that $\mathbb{Q}^{(\alpha,\beta,\gamma)}$ is equivalent to $\mathbb{P}$ if and only if $0 < \alpha, \beta, \gamma < 1$ and $\alpha+\beta+\gamma = 1$. Hence, the expectation of T_1 under $\mathbb{Q}^{(\alpha,\beta,\gamma)}$ is given by $\mathbb{E}_{\mathbb{Q}^{(\alpha,\beta,\gamma)}}[T_1] = \alpha(1+d)+\beta(1+m)+\gamma(1+u)$, which is a weighted average of $\{1+d, 1+m, 1+u\}$ with non-vanishing weights so that $1+d < \mathbb{E}_{\mathbb{Q}^{(\alpha,\beta,\gamma)}}[T_1] < 1+u$ follows. Furthermore, since the T_i are i.i.d. and since $S_n = S_0 \prod_{i=1}^n T_i$ we have that $\tilde{S}_n$ is a $\mathbb{Q}^{(\alpha,\beta,\gamma)}$ martingale if and only if $\mathbb{E}_{\mathbb{Q}^{(\alpha,\beta,\gamma)}}[T_1] = 1+r$. This is possible for $1+d < 1+r < 1+u$.

2. Derive conditions for martingale measures $\mathbb{Q}$.

   By using the previous results we have that $\mathbb{E}_{\mathbb{Q}^{(\alpha,\beta,\gamma)}}[T_1] = \alpha(1+d)+\beta(1+m)+\gamma(1+u) = (1+r)$, so that

   $$\alpha\frac{1+d}{1+r} + \beta\frac{1+m}{1+r} + \gamma\frac{1+u}{1+r} = 1\,.$$

   Furthermore, we have $1 = \alpha+\beta+\gamma$. Hence, $\alpha(d-r)+\beta(m-r)+\gamma(u-r) = 0$, along with $0 < \alpha, \beta, \gamma < 1$ and $\alpha+\beta+\gamma = 1$ is the condition we are looking for.

3. Derive that a viable market in this model is not complete.

   Above we have found a system with two equations and three variables (along with six constrains), i.e. in general this admits more than one solution (given that the market is viable).

Exercise 3: Incomplete markets

Denote by $(S_n)_{n=0,\ldots,N}$ the price vector in a viable (but not necessarily complete) market defined on a finite probability space where each element is an atom. Suppose that the random variable h defined on the same space is attainable (recall that this means that it can be replicated by an admissible strategy).

1. Show that the price V_n at n of a derivative with payoff h can be calculated uniquely by

   $$V_n = S_n^0\,\mathbb{E}^*\left[\frac{h}{S_N^0}\,|\,\mathcal{F}_n\right]$$

   where $\mathbb{E}^*$ is the expectation with respect to any measure $\widetilde{\mathbb{P}}$ under which $(\widetilde{S}_n)$ is a martingale.

   By denoting the existing replication strategy by ϕ we have that $U_n(\phi) = U_0(\phi) + \sum_{i=1}^n \phi_i \cdot \Delta S_i$ is the value of the replication strategy at n. Furthermore, since h is replicated by the strategy

ϕ we also have $U_N = h$ almost surely and hence, $\widetilde{U}_N = \widetilde{h}$ also holds almost surely. By taking conditional expectations on both sides we end up with

$$\mathbb{E}^* \left[\frac{h}{S_N^0} \,|\, \mathcal{F}_n \right] = \mathbb{E}^* \left[\frac{U_N}{S_N^0} \,|\, \mathcal{F}_n \right] = \widetilde{U}_n \,,$$

where the last equation follows the martingale property of $\widetilde{S}$ under the measures in question along with the martingale transform proposition from the lecture course (which holds in this setting). Since $\widetilde{U}_n$ does not depend on the choice of the risk-neutral probability measure we obtain that the value of the expectations is the same for all risk-neutral probability measures.

2. Give an example of an incomplete market and of an attainable product in this market.

   The trinomial model in Exercise 2 defines an incomplete market, where e.g. a risk-less investment is an attainable product and where also a forward can still be replicated by a self-financing strategy.