Exercises Martingales in Financial Mathematics: The CRR model (Solutions)

Week 2, 2025

Exercise 1: Cox Ross Rubinstein model

There is only one risky asset in the CRR model with price S_n at n until N along with a risk-less asset with risk-free interest rate r for every time period, i.e. $S_n^0 = (1+r)^n$. The risky asset is modelled as follows. Between two consecutive periods the price changes by a factor $1+a$ or $1+b$

$$S_{n+1} = \begin{cases} S_n\,(1+a) \\ S_n\,(1+b) \end{cases}$$

where $-1 < a < b$.

Suppose that the initial stock price is given by S_0 and define the probability space $(\Omega,\, \mathcal{F},\, \mathbb{P})$ with $\Omega = \{1+a,\, 1+b\}^N$, $\mathcal{F} = \mathcal{P}(\Omega)$, and $\mathbb{P}$ a probability measure such that $\mathbb{P}(\omega) > 0$ for every atom ω. For $n = 1, \ldots, N$ the σ-algebra $\mathcal{F}_n$ is generated by the random variables $S_1, \ldots, S_n$, i.e. $\mathcal{F}_n = \sigma(S_1, \ldots, S_n)$ ($\mathcal{F}_0 = \{\Omega, \varnothing\}$). We define the random variables $T_n = S_n/S_{n-1}$, with possible values $1+a$ and $1+b$, respectively.

1. Show that in order to end up with a viable market it is necessary that $r \in]a,\, b[$.

   The market is viable if and only if there is a probability measure $\mathbb{Q}$ equivalent to $\mathbb{P}$ under which the discounted asset price sequences are martingales. Hence, under $\mathbb{Q}$ we have $\mathbb{E}^{\mathbb{Q}}[\widetilde{S}_{n+1} \,|\, \mathcal{F}_n] = \widetilde{S}_n$. Thus, $\mathbb{E}^{\mathbb{Q}}[T_{n+1}\,(1+r)^{-1} \,|\, \mathcal{F}_n] = 1$. Since T_i takes values in $\{1+a,\, 1+b\}$ where both values have strictly positive probability, we end up with the result.

2. Find examples for violation of the assumption of absence of arbitrage if $r \notin]a,\, b[$.

   Consider the case $r \leq a$. We can borrow the amount S_0 at $t = 0$ and buy a share of the risky asset. At N we sell the risky asset and we pay back $S_0\,(1+r)^N$. The gain is $S_N - S_0\,(1+r)^N$ being strictly positive with a non-vanishing probability since the events $\{S_N = S_0(1+b)^N\}$ etc. have strictly positive probability. Furthermore, $S_N \geq S_0(1+a)^N \geq S_0(1+r)^N$. Hence, we end up with an arbitrage opportunity. Analogous considerations yield an arbitrage possibility for the case $r \geq b$ (arbitrage is obtained by selling the risky asset etc.).

3. Now let $r \in]a,\, b[$ and denote $p^* = (b-r)/(b-a)$. Show that $(\widetilde{S}_n)$ is a martingale under $\mathbb{Q}$ if and only if the random variables T_1, T_2, ..., T_N are i.i.d. and $\mathbb{Q}[T_1 = 1+a] = p^* = 1 - \mathbb{Q}(T_1 = 1+b)$.

   Note that for the given deterministic S_0 we have $S_n = S_0 \Pi_{i=1}^n T_i$ and $\mathcal{F}_n = \sigma(T_1, \ldots, T_n)$ for $n \geq 1$. For T_i being i.i.d. Bernoulli random variables with $\mathbb{Q}[T_i = 1+a] = p^*$ we have $\mathbb{E}^{\mathbb{Q}}[T_{n+1}|\mathcal{F}_n] = \mathbb{E}^{\mathbb{Q}}[T_{n+1}] = 1+r$. Hence, $\mathbb{Q}$ is a martingale measure.

Conversely, if there exists a probability measure $\mathbb{Q}$ such that $(\widetilde{S}_n)$ is a martingale under $\mathbb{Q}$ we have that $\mathbb{E}^{\mathbb{Q}}[T_{n+1} \,|\, \mathcal{F}_n] = 1 + r$. Hence, we can write

$$1 + r = \mathbb{E}^{\mathbb{Q}}[T_{n+1} \,|\, \mathcal{F}_n] = (1+a)\, \mathbb{E}^{\mathbb{Q}}[\mathbb{I}_{T_{n+1}=1+a} \,|\, \mathcal{F}_n] + (1+b)\, \mathbb{E}^{\mathbb{Q}}[\mathbb{I}_{T_{n+1}=1+b} \,|\, \mathcal{F}_n] .$$

Furthermore, we have

$$\mathbb{E}^{\mathbb{Q}}[\mathbb{I}_{T_{n+1}=1+a} \,|\, \mathcal{F}_n] + \mathbb{E}^{\mathbb{Q}}[\mathbb{I}_{T_{n+1}=1+b} \,|\, \mathcal{F}_n] = 1 .$$

Thus, we conclude that $\mathbb{E}^{\mathbb{Q}}[\mathbb{I}_{T_{n+1}=1+a} \,|\, \mathcal{F}_n] = \mathbb{Q}[T_{n+1} = 1 + a \,|\, \mathcal{F}_n] = p^*$. By recursion we obtain for all $x_i \in \{1+a\,, 1+b\}$

$$\mathbb{Q}(T_1 = x_1\,, \ldots\,, T_n = x_n) = \prod_{i=1}^{n} p_i$$

where $p_i = p^*$ if $x_i = 1 + a$ and $p_i = 1 - p^*$ if $x_i = 1 + b$. This proves that the T_i are independent and identically distributed under $\mathbb{Q}$ and verifies that $\mathbb{Q}(T_i = 1 + a) = p^*$.

4. Derive that the viable market obtained in 3 is complete (see Slides 52 and 53) and give a formula for the price of a claim with payoff H in the form of a conditional expectation with respect to $\mathbb{Q}$.

   We have seen that $(\widetilde{S}_n)$ is a $\mathbb{Q}$-martingale determining the law of $(T_1, T_2, \ldots, T_N)$ under $\mathbb{Q}$, and hence, the measure $\mathbb{Q}$ itself in a unique way. Hence, the market is viable and complete.

   In view of that the price of a claim is given by $p(n,\, H) = (1+r)^{-(N-n)}\mathbb{E}^{\mathbb{Q}}[H \,|\, \mathcal{F}_n]$.

Exercise 2: Pricing of options

Continue with the notation and assumptions in the previous exercise. Furthermore, denote by C_n (P_n, respectively) the value at n of a European call (put) option with strike K and maturity N (both being written on the risky asset).

1. "Rediscover" the European put-call parity based on Point 4 of the previous exercise, i.e. derive

$$C_n - P_n = S_n - K\,(1+r)^{-(N-n)} .$$

   In view of the previous exercise we have

$$\begin{aligned} C_n - P_n &= (1+r)^{-(N-n)}\mathbb{E}^{\mathbb{Q}}[(S_N - K)^+ - (K - S_N)^+ \,|\, \mathcal{F}_n] \\ &= (1+r)^{-(N-n)}\mathbb{E}^{\mathbb{Q}}[S_N - K \,|\, \mathcal{F}_n] \\ &= S_n - K(1+r)^{-(N-n)} , \end{aligned}$$

   where for the last equality we have used that $(\widetilde{S}_n)$ is a martingale under $\mathbb{Q}$.

2. Show that $C_n = c(n,\, S_n)$, where c is a function which can be expressed with the help of K, a, b, r and p^*.

   We know that $S_N = S_n\, T_{n+1} \cdots T_N$. Hence,

$$C_n = (1+r)^{-(N-n)}\, \mathbb{E}^{\mathbb{Q}}\left[(S_n\, T_{n+1} \cdots T_N - K)^+ \,|\, \mathcal{F}_n\right] .$$

Since under $\mathbb{Q}$, the variables $T_{n+1}, \cdots, T_N$ are independent of $\mathcal{F}_n$ and furthermore, S_n is $\mathcal{F}_n$-measurable, we conclude by using one of the properties of conditional expectations that $C_n = c(n, S_n)$ where c is a function being defined by

$$\begin{aligned} c(n,x) &= \frac{\mathbb{E}^{\mathbb{Q}}\left(x\Pi_{i=n+1}^{N} T_i - K\right)^+}{(1+r)^{N-n}} \qquad (1) \\ &= \frac{1}{(1+r)^{N-n}} \sum_{j=0}^{N-n} \frac{(N-n)!}{(N-n-j)!j!} p^{*j}(1-p^*)^{N-n-j} \left(x(1+a)^j(1+b)^{N-n-j} - K\right)_+ . \end{aligned}$$

3. Show that

$$c(n,\, x) = \frac{p^*}{1+r} c(n+1,\, x\,(1+a)) + \frac{1-p^*}{1+r} c(n+1,\, x\,(1+b))\,,\ n = 0, \ldots, N-1\,.$$

By conditioning on $(T_{n+2}, \ldots, T_N)$ in (1) we obtain

$$\frac{c(n,x)}{(1+r)^{-(N-n)}} = p^* \mathbb{E}^{\mathbb{Q}}\left(x(1+a)\Pi_{i=n+2}^{N} T_i - K\right)^+ + (1-p^*)\mathbb{E}^{\mathbb{Q}}\left(x(1+b)\Pi_{i=n+2}^{N} T_i - K\right)^+ .$$

4. Show that the perfect hedging strategy of a European call at n is defined by a quantity $H_n = \Delta(n, S_{n-1})$ representing the investment in the risky asset, where the Δ is a function, which can be expressed in terms of the function c.

Denote by H_n^0 the quantity invested in risk-less asset in the replication portfolio of a call. Then we have

$$H_n^0(1+r)^n + H_n S_n = c(n, S_n)\,.$$

Since H_n^0 and H_n are $\mathcal{F}_{n-1}$-measurable, they are only functions of $S_1, \ldots, S_{n-1}$ and S_n is equal to $S_{n-1}(1+a)$ or $S_{n-1}(1+b)$ so that

$$\begin{aligned} H_n^0(1+r)^n + H_n S_{n-1}(1+a) &= c(n, S_{n-1}(1+a))\,, \\ H_n^0(1+r)^n + H_n S_{n-1}(1+b) &= c(n, S_{n-1}(1+b))\,. \end{aligned}$$

By subtraction we end up with

$$\Delta(n,x) = \frac{c(n,x(1+b)) - c(n,x(1+a))}{x(b-a)}\,.$$