Martingales in Financial Mathematics

Remarks on the theory of arbitrage for continuous-time models

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Very roughly speaking the Fundamental Theorem of Asset Pricing states that, essentially, a model of a financial market is free of arbitrage if and only if there is a probability measure $\mathbf{Q}$, equivalent to the original real-world measure $\mathbf{P}$ (i.e. $\mathbf{P}$ and $\mathbf{Q}$ vanish on the same events), such that the discounted asset price processes are martingales under $\mathbf{Q}$.

Measure $\mathbf{Q}$ is then called equivalent martingale measure. In this case taking discounted expectations with respect to $\mathbf{Q}$ in order to price contingent claims yields the arbitrage-free pricing rules, where $\mathbf{Q}$ runs through all equivalent martingale measures.

If furthermore $\mathbf{Q}$ is unique, then the discounted expected (with respect to $\mathbf{Q}$) payoff yields the unique arbitrage-free price.

This theorem was proved by Harrison and Krebs (1979)/Harrison and Pliska (1981) for the case where the underlying probability space is finite (we have discussed that in the first lectures).

The Fundamental Theorem of Asset pricing has different formulations depending on the level of generality. It paves the way for using martingale methods in mathematical finance.

It turned out that in the continuous time setting, a very simple no-arbitrage condition does not guarantee the existence of an equivalent martingale measure and not even of a more general equivalent local martingale measure.

Due to this fact certain, at first glance economically not completely unreasonable modifications of the no-arbitrage property have been introduced by Delbaen and Schachermayer 1994 in order to guarantee at least the existence of an even more general equivalent σ-martingale measure, see Delbaen and Schachermayer 1998.

Recent work by Platen and other researchers indicate, however, that in continuous time market models the modifications due to Delbaen and Schachermayer exclude some models with economically very interesting properties, which are not excluded under other definitions of absence of arbitrage.

Continuous time financial markets: Basics

Uncertainty in the market is usually modelled by a *filtered probability space* $(\Omega, \mathfrak{F}, (\mathfrak{F}_t)_{t\in\mathbb{T}}, \mathbf{P})$, where $\mathbb{T} = [0, T]$, for fixed $T > 0$, or $\mathbb{T} = \mathbb{R}_+ = [0, \infty)$.

In view of the possibility to embed different time sets into $\mathbb{R}_+$, the following definitions and results are stated for $\mathbb{R}_+$.

The *filtration* $\mathfrak{F}_t$, $t \geq 0$, being an increasing family of sub-σ-algebras, is assumed to satisfy the "usual conditions" of

— *right continuity* ($\mathfrak{F}_t = \bigcap_{s>t} \mathfrak{F}_s$) and

— *completeness* ($\mathfrak{F}_0$ contains all $\mathbf{P}$-null sets of $\mathfrak{F}$).

The role of the filtration is to describe the information available at any time t.

A *stochastic process* X on $(\Omega, \mathfrak{F}, (\mathfrak{F}_t)_{t\geq 0}, \mathbf{P})$ is a collection of $\mathbb{R}$- or $\mathbb{R}^n$-valued random variables (vectors) $X = (X_t)_{t\geq 0}$.

The process X is said to be *adapted* if X_t is $\mathfrak{F}_t$ measurable for each t, i.e. X_t is known if $\mathfrak{F}_t$ is known.

Processes with sample paths being a.s. right-continuous with left limits are called *càdlàg*.

A map $\tau : \Omega \mapsto [0, \infty]$ is a *stopping time* if $\{\tau \le t\} \in \mathfrak{F}_t$ for all $t \ge 0$. The *stopping time* σ*-algebra* $\mathfrak{F}_\tau$ is defined as

$$\mathfrak{F}_\tau = \{A \in \mathfrak{F} : A \cap \{\tau \le t\} \in \mathfrak{F}_t \text{ for all } t \ge 0\} .$$

Intuitively, $\mathfrak{F}_\tau$ represents the events known at time τ.

For a stochastic process X, and a stopping time τ, the process X^τ defined by $X_t^\tau = X_{t \wedge \tau}, t \ge 0$, is the *stopped process* (at τ), where $a \wedge b = \min(a, b)$ for a, $b \in \mathbb{R}$.

Of particular interest in finance are *martingales*, recall that these are real-valued, adapted stochastic processes $M = (M_t)_{t\geq 0}$, where M_t is integralbe for all $t \geq 0$, and $\mathbf{E}(M_t|\mathfrak{F}_s) = M_s$ a.s. for all $s \leq t$.

An adapted process M is called a *local martingale* if there exists an a.s. increasing sequence $(\tau_k)_{k=1}^{\infty}$ of stopping times such that $\tau_k \to \infty$ a.s. and, for each $k \geq 1$, the stopped process M^{τ_k} is a martingale.

A process $(A_t)_{t\geq 0}$ is of bounded variation, if

$$\sup_{0\leq t_0<\cdots<t_n\leq t} \sum_{i=1}^{n} |A_{t_i} - A_{t_{i-1}}| < \infty \quad \text{a.s., for each } t < \infty\,.$$

An $\mathbb{R}^n$-valued process $M = (M_1, \ldots, M_n)$ is said to be a (local) martingale if $M_1, \ldots, M_n$ are one-dimensional (local) martingales. An $\mathbb{R}^n$-valued process $A = (A_1, \ldots, A_n)$ is of bounded variation, if its components are of bounded variation.

For $\tau_1 \leq \tau_2$ being two stopping times, $[\![\tau_1, \tau_2]\!]$ denotes the set

$$\{(t, \omega) : t \in \mathbb{T}, \omega \in \Omega \text{ and } \tau_1(\omega) \leq t \leq \tau_2(\omega)\}$$

(other stochastic intervals are defined in an analogous way).

The σ-algebra generated by all stochastic intervals of the form $[\![0, \tau]\!]$ where τ is a stopping time, is called the *predictable* σ-algebra.[a] The predictable σ-algebra is denoted by $\mathfrak{P}$. Any process which is measurable with respect to $\mathfrak{P}$ is called *predictable*. The σ-algebra $\mathfrak{P}$ on $\Omega \times \mathbb{R}_+$ is also generated by the *left*-continuous $(\mathfrak{F}_t)$-adapted processes.

a. To be precise, when $\mathfrak{F}_0$ is not trivial, we also have to include the sets of the form $\{0\} \times A$, where A runs through $\mathfrak{F}_0$.

Continuous time financial markets: Asset prices

The process of traded asset prices $S = (S_{t0}, S_{t1}, \ldots, S_{tn})_{t\geq 0}$, where $(S_{t0})_{t\geq 0}$ often stands for a money market account, is usually assumed to be an $\mathbb{R}^{n+1}$-valued semimartingale defined over and adapted to the filtered probability space $(\Omega, \mathfrak{F}, (\mathfrak{F}_t)_{t\geq 0}, \mathbf{P})$.

In general, a n-dimensional càdlàg process X, is called a *semimartingale* if it admits a decomposition

$$X_t = X_0 + A_t + M_t\,, \quad t \geq 0\,, \tag{1}$$

where $(A_t)_{t\geq 0}$ is a n-dimensional adapted process of *bounded variation*, $(M_t)_{t\geq 0}$ is a n-dimensional local martingale, and $A_0 = M_0 = 0$.

A semimartingale X is a special semimartingale if a decomposition of this form exists with A being also predictable. If such a decomposition exists, it is unique. In particular this is the case for continuous semimartingales.

The use of semimartingales in mathematical finance is justified by several reasons. The main among them might be, firstly, that this class is wide enough, and secondly, that for semimartingales the theory of stochastic integration is well developed which suits fine for the construction of arbitrage theory.

For formulating the Fundamental Theorem of Asset Pricing in the continuous time higher-dimensional setting, integrals with respect to vector-valued semimartingales have to be constructed (for integrable H, assumed to be predictable). The first quite tricky construction [a] of the sufficiently general so-called *vector* (not vector-valued) *stochastic integrals* denoted by

$$H \cdot X = (H \cdot X)_t = \int_0^t \langle H_s, dX_s \rangle \,, \quad t \geq 0 \,,$$

has been provided by Jacod (1980). An explicit approach to the vector stochastic integral and a discussion of its application in the context of the (First and Second) Fundamental Theorem of Asset Pricing has been provided by Cherny and Shiryaev (2002), based on (1998).

a. To get closedness of the space of integrals is the difficult point.

If the *componentwise stochastic integral* $\sum_{i=1}^{n}(H_i \cdot X_i)_t$, $t \geq 0$, of one-dimensional stochastic integrals exists, see e.g. Barndorff-Nielsen and Shiryaev (2015) or Jacod and Shiryaev (2003), then the vector stochastic integral also exists and the integrals coincide. However, it may happen that the componentwise stochastic integral does not exist, while the vector stochastic integral is well defined.

For an example, showing the necessity of using vector stochastic integral rather than componentwise in the context of the Fundamental Theorem of Asset Pricing, see again Cherny and Shiryaev (2002).

A semimartingale $X = (X_{t1}, \ldots, X_{tn})_{t\geq 0}$ is a σ*-martingale* if there exists an $\mathbb{R}^n$-valued local martingale M and a process $H = (H_{t1}, \ldots, H_{tn})_{t\geq 0}$ such that for each i, H_i is M_i-integrable and $X_i = X_{0i} + H_i \cdot M_i$.

Note that every càdlàg martingale is a local martingale (take $\tau_n = n$). Furthermore, a local martingale is a σ-martingale. The converse is not true. The archetype of a local martingale which fails to be a martingale is the inverse Bessel (3) process, see e.g. Delbaen and Schachermayer Ex. 7.2.8. For an example of a σ-martingale not being a local martingale, see Émery (1980).

A *self-financing strategy* π is a pair (x, H), where $x \in \mathbb{R}$ and predictable process $H = (H_{t1}, \ldots, H_{tn})_{t \geq 0}$ is $\tilde{S}$-integrable, i.e. there exists a vector stochastic integral $H \cdot \tilde{S}$, where $\tilde{S}$ stands for the discounted price processes of the risky assets, i.e. $S_{t0} = B_t$ (assumed to be a.s. strictly positive), $t \geq 0$,

$$\tilde{S}_t = \left(\frac{S_{t1}}{B_t}, \ldots, \frac{S_{tn}}{B_t} \right), \quad t \geq 0 \, .$$

The discounted capital process of the strategy $\pi = (x, H)$ is given by

$$V_t^\pi = x + (H \cdot \tilde{S})_t \,, \quad t \geq 0 \,,$$

where $(H \cdot \tilde{S})_t$, $t \geq 0$, represents the discounted return of the strategy π over time t (note in order to simplify the notation V stands here for the discounted return).

Note that for every $t \geq 0$ we also have

$$V_t^\pi = \langle H_t, \tilde{S}_t \rangle + H_{t0} \,,$$

where H_{t0} is the amount invested in asset 0 at time t (where $B_t / B_t = 1$ for every $t \geq 0$). The cumulative (discounted) costs up to time t incurred by a (not necessarily self-financing) strategy is

$$C_t = V_t^\pi - (H \cdot \tilde{S})_t \,.$$

In the self-financing case it is $C_t = x$ for every t, i.e. a self-financing strategy is completely described by x and H, determining H_{0t} for every $t \geq 0$.

Arbitrage and all that

Definition 1. *A strategy* $\pi = (x, H)$ *realises* arbitrage *if*

(i) $x = 0$,

(ii) there exists a constant $a \geq 0$ *such that*

$$\mathbf{P}(V_t^\pi \geq -a \text{ for all } t \geq 0) = 1\,,$$

(iii) the limit $V_\infty^\pi = \lim_{t\to\infty} V_t^\pi$ *exists* $\mathbf{P}$*-a.s.*,

(iv) $V_\infty^\pi \geq 0$ $\mathbf{P}$*-a.s.*,

(v) $\mathbf{P}(V_\infty^\pi > 0) > 0$.

A model satisfies the No-Arbitrage *condition (notation* NA*) if such a strategy does not exist.*

Definition 2. *A sequence of strategies* $\pi_k = (x_k, H_k)$, $k \geq 0$, *realises* free lunch with vanishing risk *if for all* $k \geq 0$,

(i) $x_k = 0$,

(ii) there exists a constant $a_k > 0$ *such that* $\mathbf{P}(V_t^{\pi_k} \geq -a_k$ *for all* $t \geq 0) = 1$,

(iii) the limit $V_\infty^{\pi_k} = \lim_{t\to\infty} V_t^{\pi_k}$ *exists* $\mathbf{P}$*-a.s.* ,

(iv) $V_\infty^{\pi_k} \geq -\frac{1}{k}$ $\mathbf{P}$*-a.s.* ,

(v) there exist constants δ_1, $\delta_2 > 0$ *such that* $\mathbf{P}(V_\infty^{\pi_k} > \delta_1) > \delta_2$ *(*δ_1 *and* δ_2 *independent of* k*).*

A model satisfies the No Free Lunch with Vanishing Risk *condition (notation* NFLVR*) if such a sequence of strategies does not exist.*

The Fundamental Theorems of Asset Pricing

Theorem 1 ((First) Fundamental Theorem of Asset Pricing)**.** *The following assertions are equivalent for an $\mathbb{R}^n$-valued semimartingale model $\tilde{S}$, of a financial market.*

(i) There is a probability measure $\mathbf{Q}$ equivalent to $\mathbf{P}$ such that $\tilde{S}$ is a σ-martingale under $\mathbf{Q}$,

(ii) $\tilde{S}$ satisfies NFLVR.

Due to a result by Ansel and Stricker (1994) we have that if the components of $\tilde{S}$ are nonnegative, NFLVR is equivalent to the existence of an equivalent local martingale measure (i.e. a measure $\mathbf{Q}$ under which $\tilde{S}$ is a local martingale).

Arbitrage results for certain models including transaction costs can be found e.g. in Kabanov and Safarian (2009).

The so-called Second Fundamental Theorem of Asset Pricing roughly tells us that an arbitrage-free model is complete if the equivalent martingale measure is unique, where, informally speaking, completeness means a possibility to hedge “any” contingent claim.

It should be pointed out that all models employed in “classical mathematical finance” are free of arbitrage, but only a few of them are complete, while most of the models are incomplete.

Definition 3. *A model of a market described by the semimartingale* $\tilde{S}$ *is* complete *if for any bounded* $\mathfrak{F}$*-measurable random variable* Y *one can find a strategy* π *such that*

(i) for some constants a *and* b

$$\mathbf{P}(a \leq V_t^\pi \leq b \text{ for all } t \geq 0) = 1\,,$$

(ii) the limit $V_\infty^\pi = \lim_{t\to\infty} V_t^\pi$ *exists* $\mathbf{P}$*-a.s.*,

(iii) $Y = V_\infty^\pi$ $\mathbf{P}$*-a.s.*.

Definition 4. *Let* $\tilde{S} = (\tilde{S}_{t1}, \ldots, \tilde{S}_{tn})_{t \geq 0}$ *be a semimartingale given on a filtered probability space* $(\Omega, \mathfrak{F}, (\mathfrak{F}_t)_{t \geq 0}, \mathbf{Q})$*. One says that a local martingale* M *given on this probability space admits* $\tilde{S}$-representation *if one can find an integrable, predictable process* $H = (H_{t1}, \ldots, H_{tn})_{t \geq 0}$ *such that for all* $t \geq 0$

$$M_t = M_0 + (H \cdot \tilde{S})_t \,, \quad \text{a.s.}$$

Theorem 2 (Second Fundamental Theorem of Asset Pricing)**.**

(a) Assume that the family of equivalent σ-martingale measures is nonempty.

Then the following conditions are equivalent

(i) the model described by the semimartingale $\tilde{S}$ is complete,

(ii) the family of equivalent σ-martingale measures consists of a single measure,

(iii) in the family of equivalent σ-martingale measures exists a measure $\mathbf{Q}$ such that any local martingale M, admits the $\tilde{S}$-representation (with respect to this measure).

(b) Assume that the components of $\tilde{S}$ are nonnegative and that the family of equivalent local martingale measures is nonempty. Then the following conditions are equivalent

- *(i) the model described by the semimartingale $\tilde{S}$ is complete,*
- *(ii) the family of equivalent local martingale measures consists of a single measure,*
- *(iii) in the family of equivalent local martingales exists a measure $\mathbf{Q}$ such that any local martingale M admits the $\tilde{S}$-representation (with respect to this measure).*

Hence, in case of semimartingales with non-negative components, we in particular have that the existence of exactly one local martingale measure ensures completeness of the model. It should be emphasised that Theorem 2 does not hold if the vector stochastic integrals are replaced by the componentwise stochastic integrals. Furthermore, a model can be complete, whereas the family of equivalent σ-martingales is empty.

Risk-neutral valuation

The above results can be applied to the so-called *risk-neutral* valuation.

E.g. consider a finite time horizon $T > 0$ and a $\mathfrak{F}_T$-measurable non-negative random variable C, a so-called (non-discounted) *contingent claim*, i.e. a payoff at time T, but the amount to be paid may depend on the whole information contained in $\mathfrak{F}_T$. Assume that $Y = B_T^{-1}C$ is $\mathbf{Q}$-integrable for some $\mathbf{Q}$ being equivalent to $\mathbf{P}$ such that all components of $\tilde{S}$ are $\mathbf{Q}$-martingales, and put

$$P_t = B_t \mathbf{E}_{\mathbf{Q}}(B_T^{-1}C|\mathfrak{F}_t)\,, \quad t \in [0,T]. \tag{2}$$

If we consider $(P_t)_{t\in[0,T]}$ to be the price process of an asset, then the market extended with this asset is still free of arbitrage, since the discounted price process of this asset is a $\mathbf{Q}$-martingale.

So, $(P_t)_{t\in[0,T]}$ (satisfying $P_T = C$) is a candidate for a "fair" price process of C.

However, this definition of "fair" depends on the choice of the equivalent martingale measure.

If there is an equivalent martingale measure the corresponding market satisfies NFLVR implying NA.

Furthermore, assume that in this arbitrage free market the equivalent martingale measure is unique in the class of equivalent local martingale measures.

Then, for "suitably defined" (e.g. as in the Second Fundamental Theorem of Asset Pricing) contingent claims, there exists a self-financing[a] *dynamic hedging strategy* consisting of continuously trading in the assets (including the bond), having value

$$B_t \mathbf{E}_{\mathbf{Q}}(B_T^{-1} C | \mathfrak{F}_t) \, , \quad \text{for all } t \in [0, T] \, .$$

a. Recall that this means that there is only an initial investment.

In general, contingent claims which can be replicated in this way are often called attainable.

The advantage of complete markets stems from the fact that it allows the pricing and hedging of contingent claims in a preference-independent way.

However, completeness is a quite delicate property which is typically destroyed as soon as one considers even minor modifications of a basic complete model.

For instance, we know that the classical Black–Scholes model based on a one-dimensional geometric Brownian motion, is complete, but becomes incomplete if the volatility is influenced by a second stochastic factor or if one adds a jump component to the model.

If one insists on the preference-free approach under incompleteness, one can study the range of possible prices for C which are consistent with the absence of arbitrage in a market.

Another approach is to introduce subjective criteria according to which strategies the martingale measures are chosen.

Hence, for a given semimartingale model satisfying NFLVR, the problem of pricing financial derivatives essentially boils down to choosing an appropriate equivalent martingale measure and then computing the conditional expectations

$$B_t \mathbf{E}_{\mathbf{Q}}(B_T^{-1} C | \mathfrak{F}_t)\,, \quad \text{for all } t \in [0, T]\,.$$

During more than the last twenty-five years many different strategies for choosing an appropriate martingale measure have been developed, e.g. in the sense of minimising a distance from the measure $\mathbf{P}$ (entropic distance, L^2-distance, general f-distance, Hellinger distance, etc., or in the sense of constructing the simplest possible measure, e.g. steaming from the Esscher transform).

From the practitioners point of view, the choice of this measure should be the result of a calibration of the model to the market price of plain vanilla options (the corresponding inverse problems are often ill-posed so that regularization should be used).

Applying an incomplete market model yields for many claims that there are no perfect hedging strategies by investing solely in the underlying and the cash-bond.

If certain derivatives of underlyings are traded on the market, it is possible to hold a dynamic portfolio containing these instruments, typically consisting of cash-bonds, a position in the univariate underlying, and a collection of calls and puts. If one allows this dynamic trading in options, the market can be completed in some situations by adding a suitable collection of derivatives, e.g. European calls or variance swaps, to the liquidly traded underlying asset.