Martingales in Financial Mathematics

Michael Schmutz

Model extensions

Parts of the presentation are based on a presentation by

Albert N. Shiryaev

Searching for adequate stochastic models which describe dynamics of the prices

As on finite probability spaces the theory on the fundamental theorems of asset pricing in continuous time lead to an important but quite abstract theory about "absence of arbitrage" and about complete markets, a notion being closely related to hedging.

However, one of the key topics in both theory of mathematical finance and practice of financial trading consists in realizing *which process* simulate the prices S respectively $\tilde{S}$.

In the discrete case we have considered e.g. the trinomial model (incomplete market) and in particular the CRR (complete market) model.

In the continuous case we have already discussed the Black–Scholes (complete market) model for the *concrete* behaviour of share price processes.

Recall that the Black–Scholes model

$$S_t = S_0 e^{(\mu - \frac{1}{2}\sigma^2)t + \sigma B_t}, \quad \text{i.e.} \quad dS_t = S_t(\mu dt + \sigma dB_t)$$

with a *constant* volatility σ has the weakness that really observable data tells us that σ is NOT a constant (volatility skew, volatility smile, non-constant volatility surface).

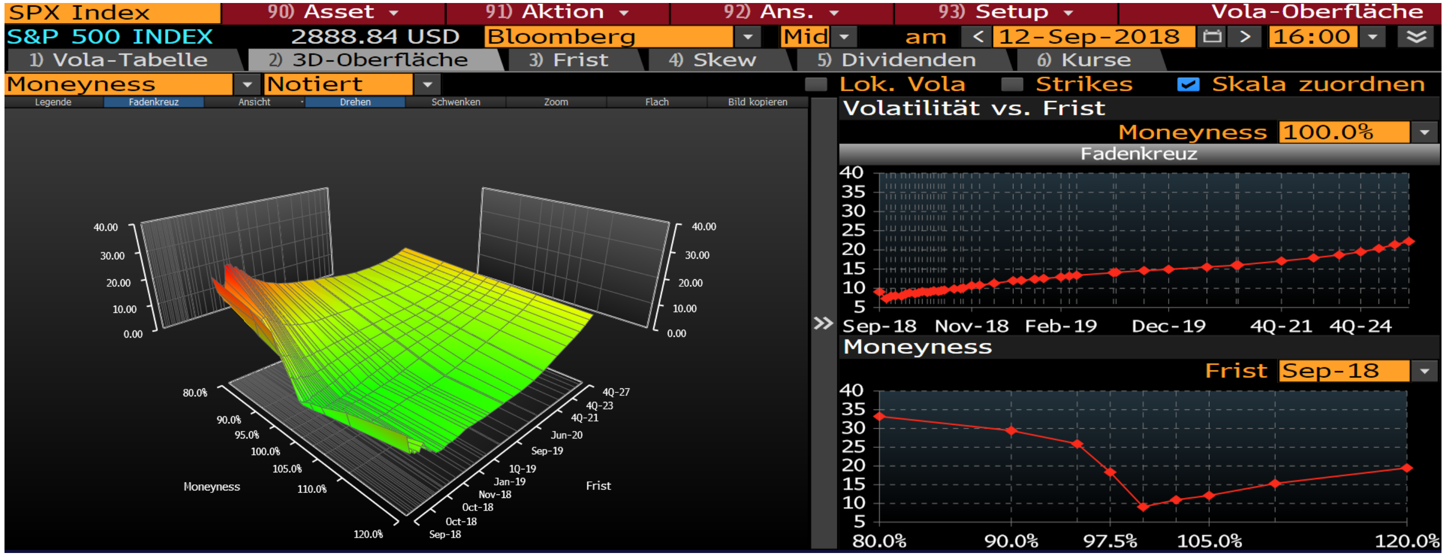

Figure 1 – *Source: Bloomberg*

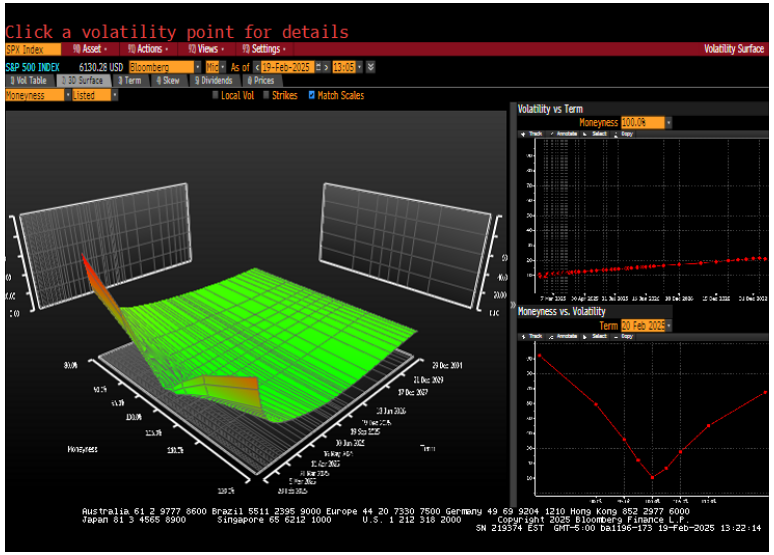

Figure 2 – *Source: Bloomberg*

1st Correction $\sigma \rightsquigarrow \sigma(t)$ (R. Merton, 1973)

2nd Correction $\sigma(t) \rightsquigarrow \sigma(t, S_t)$ (B. Dupire, 1994) .

More concretely, in the framework of models with nonstochastic volatility, one can obtain smile effects by assuming $\sigma = \sigma(t, x)$ for a deterministic function σ.

Then consider

$$dS_t = \mu S_t dt + S_t \sigma(t, S_t) dB_t \, ,$$

for which the advanced theory of existence and uniqueness of strong and weak solutions is developed.

Annoying circumstance: The prices and volatility turn out to be perfectly correlated and this contradicts statistical observations showing that the correlation should be negative but not equal to -1.

For the intuition. . .

Figure 3 – *Source: Bloomberg*

Also natural is to go further, i.e. to assume that volatility depends (and reflects) not only on t and S_t but also on all preceding values S_u, $u \leq t$ (the history), i.e. consider

$$dS_t = S_t \mu dt + S_t \sigma(t; S_u, u \leq t) dB_t \,,$$

$\sigma(t; x_u, u \leq t)$.

— markets are complete

— study of these models is not advanced because of analytical difficulties (some results concerning existence of solutions are given in Ch. 4 in R. Sh. Liptser
A. N. Shiryaev, Statistics of Random Processes (2001)).

Classic stochastic volatility models

Assume the volatility to be itself “volatile”, i.e. that the volatility is generated by a source of randomness which is different from the driving BM B.

In the framework of “Brownian models” another Brownian motion $\tilde{B}$, with

$$d\langle B, \tilde{B}\rangle_t = \rho \, dt \, ,$$

for suitable $\rho \in (-1, 1)$.

As to the volatility $\sigma(t)$, it is convenient to assume that

$$\sigma(t) = f(Y_t) \, ,$$

where $f(y)$ is a nonnegative function, e.g. e^y or $\sqrt{|y|}$ and Y is a diffusion which satisfies

$$dY_t = a(t, Y_t)dt + b(t, Y_t)d\tilde{B}_t \, .$$

Or even one could assume that Y belongs to the larger class of Itô-processes

$$dY_t = a(t,\omega)dt + b(t,\omega)d\tilde{B}_t \, ,$$

with $a(t,\omega)$ and $b(t,\omega)$ being $\mathcal{F}_t$-measurable functions for every t and such that a.s.

$$\int_0^t |a(s,\omega)|ds < \infty \, , \quad \int_0^t b^2(s,\omega)ds < \infty \, , \quad t > 0 \, .$$

Some difficulty arises: The coefficients $a(t,\omega)$ and $b(t,\omega)$ must be "adjusted" to the behavior of volatility, which is not directly observable.

Nevertheless, indirect observations can allow one to make important conclusions about properties of volatility itself. Perhaps most important: mean reversion, that is, return of the process towards the mean.

Simplest process with mean reversion: Ornstein-Uhlenbeck process satisfying

$$dY_t = (\alpha - \beta Y_t)dt + \gamma d\tilde{B}_t\,, \quad Y_0 = y \in \mathbb{R}\,, \tag{1}$$

and $\alpha > 0$, $\beta > 0$, $\gamma > 0$.

This process takes values in $\mathbb{R}$. One can take $\sigma(t)$ equal $f(Y_t)$, where e.g. $f(y) = e^y$.

An example of models, where Y has mean reversion and is nonnegative, is given by the Cox-Ingersoll-Ross model

$$dY_t = (\alpha - \beta Y_t)dt + \gamma\sqrt{Y_t}d\tilde{B}_t\,, \quad Y_0 > 0\,,$$

$\alpha > 0$, $\beta > 0$, $\gamma > 0$.

Assume e.g. that the price S satisfies

$$dS_t = \mu S_t dt + \sigma(t) S_t dB_t \ ,$$

where $\sigma(t) = e^{Y_t}$ and Y being an OU-process (which satisfies (1)).

Then

$$d\langle S, \sigma\rangle_t = \sigma(t)^2 S_t \gamma d\langle B, \tilde{B}\rangle_t \ .$$

From this we see that to have negative correlation between prices and volatility (often observed) we must assume that the driving Brownian motions are also negatively correlated, so that $d\langle B, \tilde{B}\rangle_t = \rho dt$, where $\rho < 0.$

In practice the Heston Model is quite popular, which is based on CIR, with $f(y) = \sqrt{y}$.

(Extensions of) the Ornstein-Uhlenbeck and CIR processes are also used in order to model the short-rates $(r_t)_{t\in[0,T]}$ in simple stochastic interest rate models, where the risk-less asset is then given by $S_{t0} = \exp(\int_0^t r_s \, ds)$, $t \in [0, T]$.

The characteristic feature of these models is that they were constructed based on an integral representation of the type

$$\sigma \cdot B \, .$$

Main models based on Brownian motion

— Exponential INTEGRAL Brownian model

$$S_t = S_0 \exp\left(\int_0^t \mu_s ds + \int_0^t \sigma_s dB_s \right) \,.$$

— Exponential TIME-CHANGED Brownian model

$$S_t = S_0 \exp\left(\mu T(t) + B_{T(t)} \right) \,,$$

i.e. $\left(B_{T(t)}\right)$ is a time change in a (standard) Brownian motion.

Random Change of Time

Consider a *filtered probability space* or *stochastic basis* $(\Omega\,,\mathfrak{F}\,,(\mathfrak{F}_\theta)_{\theta\geq 0},\mathbb{P})$ (satisfying the usual conditions).

It is convenient, when defining the notion of "change of time" to distinguish between the "old" (physical, calendar) θ-time and a "new" (operational, business) t-time.

The following definition is useful if we need to construct, starting from the initial process $X=(X_\theta)_{\theta\geq 0}$ (adapted to the filtration $(\mathcal{F}_\theta)$), a new process $\hat{X}=(\hat{X}_t)_{t\geq 0}$ evolving in t-time and having certain desired properties.

Definition 1. *A family of random variables* $T = (T(t))_{t \geq 0}$ *is said to be a* random change of time*, if*

(a) $(T(t))_{t \geq 0}$ *is a nondecreasing (in the terminology of stochastic analysis–increasing), right-continuous family of* $[0, \infty]$*-valued random variables* $T(t)$*,* $t \geq 0$*;*

(b) for all $t \geq 0$ *the random variables* $T(t)$ *are stopping times with respect to the filtration* $(\mathcal{F}_\theta)_{\theta \geq 0}$*, i.e.*

$$\{T(t) \leq \theta\} \in \mathcal{F}_\theta \,, \quad t \geq 0 \,, \; \theta \geq 0 \,.$$

One out of others exceptional roles of Brownian motion is the possibility to represent a wide class of processes (in fact, semimartingales) X in the form $X \overset{\text{law}}{=} B \circ T$ (perhaps on different probability spaces) for a certain Brownian motion B and change of time T. (This is “Monroe’s theorem”, Monroe (1978)).

A particularly nice subfamily of semimartingales is given by the well-known Lévy processes.

Lévy Processes

A Lévy process $L = (L_t)_{t \geq 0}$ is a càdlàg process with independent and stationary increments, $L_0 = 0$, which is continuous in probability [a].

Kolmogorov-Lévy-Khinchin's formula (often Lévy-Khinchin's formula) for characteristic functions

$$\mathbb{E}e^{iuL_t} = \exp\left\{ t\Big(iub - \frac{u^2}{2}c + \int \big(e^{iux} - 1 - iuh(x)\big) F(dx) \Big) \right\}$$

where the classical "truncation" function is $h(x) = x\mathbb{I}_{|x| \leq 1}$ and

— $F(dx)$ is a measure on $\mathbb{R} \setminus \{0\}$ such that $\int \min(1, x^2) F(dx) < \infty$,

— $b \in \mathbb{R}$ and $c \geq 0$,

— (b, c, F) is the Lévy triplet (or the triplet of local characteristics) of L (characterizes the Lévy process).

a. i.e. for every $\varepsilon > 0$, $\lim_{s \to t} \mathbb{P}(|L_s - L_t| > \varepsilon) = 0$.

The Lévy-Itô representation for trajectories of $L = (L_t)_{t \geq 0}$ is

$$L_t = bt + L_t^c + \int_0^t \int h(x)d(\mu - \nu) + \int_0^t \int (x - h(x))d\mu \,,$$

where

— (as bt) L^c is a continuous component of L (the continuous martingale component of L), $L_t^c = \sqrt{c} W_t$, where W is a Wiener process ;

— μ is the measure of jumps: for $A \in \mathcal{B}(\mathbb{R} \setminus \{0\})$

$$\mu(\omega; (0,t] \times A) = \sum_{0<s\leq t} \mathbb{I}_A(\Delta L_s) \quad (\Delta L_s = L_s - L_{s-})\,;$$

— ν is the compensator of the measure of jumps μ :

$$\nu((0,t] \times A) = tF(A) = \mathbb{E}[\mu(\cdot, (0,t] \times A)]\,, \quad F(A) = \int_A F(dx)\,.$$

Examples of Lévy processes

— Brownian motion.

— Poisson process: A process $(N_t)_{t \geq 0}$, is a Poisson process with intensity parameter λ if it is a counting process with $N_0 = 0$ a.s., stationary and independent increments, where N_t is Poisson-distributed with parameter λt.

— compound Poisson process defined by

$$L_t = \sum_{k=0}^{N_t} \xi_k \; ,$$

where

— $(N_t)_{t \geq 0}$ is a Poisson process,

— $(\xi_k)_{k \geq 1}$ is a sequence of independent and identically distributed random variables and also independent of (N_t). Furthermore, $\xi_0 = 0$.

Hyperbolic Lévy processes

In connection with financial econometrics Hyperbolic Lévy processes are of great interest, because they model well the really observable processes $H = (H_t)_{t \geq 0}$ for many underlying financial instruments (exchange rates, stocks, etc.).

The credit of developing the theory of such processes and their applications is due to E. Halphen, O. Barndorff-Nielsen, E. Eberlein.

We will construct these processes, following mostly Chapters 9 and 12 of the monograph : O. Barndorff-Nielsen and A. Shiryaev, Change of Time and Change of Measures, 2nd Edition (2015).

For a Lévy process $(H_t)_{t\geq 0}$ we have

$$\mathbb{E}(e^{iuH_t}) = (\mathbb{E}e^{iuH_1})^t \,.$$

The properties of Lévy's processes imply that the random variable $h = H_1$ is *infinitely divisible*, i.e., for any n one can find i.i.d. r.v.'s $\xi_1, \ldots, \xi_n$ such that

$$\mathrm{Law}(h) = \mathrm{Law}(\xi_1 + \cdots + \xi_n) \,.$$

We will look for the infinitely divisible r.v.'s h having the from

$$h = \mu + \beta\sigma^2 + \sigma\varepsilon \,,$$

where ε is a standard Gaussian random variable, $\varepsilon \sim \mathcal{N}(0,1)$, $\sigma = \sigma(\omega)$ is the "volatility" (which does not depend on ε), for whose square, σ^2, we will construct a special distribution

$\mathbb{G}$IG – $\mathbb{G}$eneralized Inverse Gaussian distribution.

Strikingly, this distribution (on $\mathbb{R}_+$) is *infinitely divisible* and also the distribution of $h = \mu + \beta\sigma^2 + \sigma\varepsilon$ (on $\mathbb{R}$) is *infinitely divisible*. Due to that, there exist a Lévy processes $T = (T(t))_{t\geq 0}$ and $H^* = (H_t^*)_{t\geq 0}$ such that

$$\mathrm{Law}(T(1)) = \mathrm{Law}(\sigma^2) \quad \text{and} \quad \mathrm{Law}(H_1^*) = \mathrm{Law}(h)\,.$$

As a realization of $H^* = (H_t^*)_{t\geq 0}$ one can take

$$H_t = \mu t + \beta T(t) + B_{T(t)}\,,$$

where the “time change” $T = (T(t))_{t\geq 0}$ and the Brownian motion $B = (B_\theta)_{\theta\geq 0}$ are independent.

In the sequel, we do not distinguish between the processes H and H^*.

This process H, remarkable in many respect, bears the name

L($\mathbb{G}$H)–Generalized Hyperbolic Lévy process.

The construction of the $\mathbb{G}$IG-distributions for σ^2 is as follows.

Let $W = (W_t)_{t\geq 0}$ be a Wiener process (standard Brownian motion). For $A \geq 0$, $B > 0$ introduce

$$T^A(B) = \inf\{s \geq 0 : As + W_s \geq B\}\,.$$

The formula for the density $p_{T^A(B)}(s) = d\mathbb{P}(T^A(B) \leq s)/ds$ is well known

$$p_{T^A(B)}(s) = \frac{B}{s}\varphi_s(B - As)\,, \quad \varphi_s(x) = \frac{1}{\sqrt{2\pi s}}e^{-x^2/(2s)}\,. \qquad (2)$$

Letting $b = B^2 > 0$ and $a = A^2 \geq 0$, we find from (2) the following formula for $p(s; a, b) = p_{T^{\sqrt{a}}(\sqrt{b})}(s)$:

$$p(s; a, b) = c_1(a, b)s^{-3/2}e^{-(as+b/s)/2}\,, \quad c_1(a, b) = \sqrt{\frac{b}{2\pi}}e^{\sqrt{ab}}\,.$$

The distribution with density $p(s; a, b)$ is named

IG $=$ IG(a, b) – Inverse Gaussian distribution .

Next important step : one defines *ad hoc* the function

$$p(s; a, b, \nu) = c_2(a, b, \nu) s^{\nu - 1} e^{-(as + b/s)/2}\,, \tag{3}$$

where the parameters $a, b, \nu \in \mathbb{R}$ are chosen in such a way that $p(s; a, b, \nu)$ is a probability density on $\mathbb{R}_+$.

$$a \geq 0\,, \quad b > 0\,, \quad \nu < 0$$

$$a > 0\,, \quad b > 0\,, \quad \nu = 0$$

$$a > 0\,, \quad b \geq 0\,, \quad \nu > 0$$

$$\Downarrow$$

$$\int_0^\infty s^{\nu-1} e^{-(as+b/s)/2} ds < \infty\,.$$

It is well known that $K_\nu(y) = \frac{1}{2}\int_0^\infty s^{\nu-1} e^{-y(s+1/s)/2} ds$ is the third-kind Bessel function of order ν, which for $y > 0$ solves

$$y^2 f''(y) + y f'(y) - (y^2 + \nu^2) f(y) = 0 \,.$$

The constant in (3) has the from $c_2(a, b, \nu) = \frac{(a/b)^{\nu/2}}{2K_\nu(\sqrt{ab})}$.

The distribution on $\mathbb{R}_+$ with density

$$p(s; a, b, \nu) = \frac{(a/b)^{\nu/2}}{2K_\nu(\sqrt{ab})} s^{\nu-1} e^{-(as+b/s)/2}$$

$s > 0$, bears the name

$\mathbb{G}$IG – Generalized Inverse Gaussian distribution .

Important properties of $\mathbb{G}$IG-Distributions (for σ^2)

(A) This distribution is *infinitely divisible*.

(B) The density $p(s; a, b, \nu)$ is *unimodal* with mode

$$m = \begin{cases} b/[2(1-\nu)]\,, & \text{if } \; a = 0\,, \\ [(\nu - 1) + \sqrt{ab + (\nu-1)^2}]/a\,, & \text{if } \; a > 0\,. \end{cases}$$

(C) The *Laplace's transform* $L(\lambda) = \int_0^\infty e^{-\lambda s} p(s; a, b, \nu) ds$ is given by

$$L(\lambda) = \big(1 + \frac{2\lambda}{a}\big)^{-\nu/2} \frac{K_\nu(\sqrt{ab(1 + 2\lambda/a)})}{K_\nu(\sqrt{ab})}\,.$$

Particularly important SPECIAL CASES of $\mathbb{G}$IG-distributions are

(i) $a \geq 0, b > 0, \nu = -1/2$ in this case $\mathbb{G}\text{IG}(a, b, -1/2) = \text{IG}(a, b)$

Inverse Gaussian distribution.

Density : $p(s; a, b) = c_1(a, b)s^{-3/2}e^{-(as+b/s)/2}$, $c_1(a, b) = \sqrt{\frac{b}{2\pi}e^{\sqrt{ab}}}$,

Density of Lévy's measure : $f(y) = \sqrt{\frac{b}{2\pi}}\frac{e^{-ay/2}}{y^{3/2}}$.

(ii) $a > 0, b = 0, \nu > 0$ in this case $\mathbb{G}\text{IG}(a, 0, \nu) = \text{Gamma}(a/2, \nu)$

Gamma distribution.

Density : $p(s; a, 0, \nu) = \frac{(a/2)^\nu}{\Gamma(\nu)}s^{\nu-1}e^{-as/2}$,

Density of Lévy's measure : $f(y) = y^{-1}\nu e^{-ay/2}$.

(iii) $a > 0, b > 0, \nu = 1$

Density : $p(s; a, b, 1) = \frac{\sqrt{a/b}}{2K_1(\sqrt{ab})}e^{-(as+b/s)/2}$, is the

PH – Positive Hyperbolic distribution, or H$^+$-distribution.

Since a $\mathbb{G}$IG-distribution is infinitely divisible we have that if one takes it as the distribution of σ^2,

$$\mathrm{Law}(\sigma^2) = \mathbb{G}\mathsf{IG}\,,$$

then one can construct a *nonnegative, nondecreasing Lévy* process $T = (T(t))_{t \geq 0}$ (a subordinator) such that

$$\mathrm{Law}(T(1)) = \mathrm{Law}(\sigma^2) = \mathbb{G}\mathsf{IG}\,.$$

In the subsequent constructions, this process plays the role of

change of time, operational time, business time.

As was explained above, the next step is the construction of the (normal) log of the normalised asset price process $H = (H_t)_{t \geq 0}$.

From the variable $h = \mu + \beta\sigma^2 + \sigma\varepsilon$, where $\mathrm{Law}(\varepsilon) = \mathcal{N}(0,1)$, and from the independence of σ^2 and ε it follows that the distribution of h is a mixture of normal distributions, i.e., the density $p_h(x)$ of h is of the form

$$p_h(x) = \int_0^\infty \frac{1}{\sqrt{2\pi y}} \exp\left\{-\frac{(x-(\mu+\beta y))^2}{2y}\right\} p_{\mathbb{G}\mathsf{IG}}(y)dy\,.$$

This can be rewritten as (where $p_h(x)$ is denoted by $p^*(x;a,b,\mu,\beta,\nu)$)

$$p^*(x;a,b,\mu,\beta,\nu) = c_3(a,b,\beta,\nu)\frac{K_{\nu-1/2}(\alpha\sqrt{b+(x-\mu)^2})}{(\sqrt{b+(x-\mu)^2})^{1/2-\nu}}e^{\beta(x-\mu)}\,,$$

where $\alpha = \sqrt{a+\beta^2}$ and $c_3(a,b,\beta,\nu) = \frac{(a/b)^{\nu/2}\alpha^{\frac{1}{2}-\nu}}{\sqrt{2\pi}K_\nu(\sqrt{ab})}$.

The obtained distribution $\mathrm{Law}(h)$ with density $p^*(x; a, b, \mu, \beta, \nu)$ bears the name

Generalised Hyperbolic distribution, $\mathbb{G}\mathsf{H} = \mathbb{G}\mathsf{H}(a, b, \mu, \beta, \nu)$.

Some properties of $\mathbb{G}$H-distribution (for h)

(A^*) This distribution is *infinitely divisible*

(B^*) If $\beta = 0$, then the distribution is *unimodal* with mode $m = \mu$ (in the general case m is determined as a root of a certain transcendental equation.)

(C^*) The Lévy-Khinthchine representation is known. It contains no centered gaussian term and the Lévy measure has a (quite complicated) density.

Three important special cases of $\mathbb{G}$H-distributions

(i)' $a \geq 0, b > 0, \nu = -1/2$: In this case $\mathbb{G}\mathsf{IG}(a, b, -1/2) = \mathsf{IG}(a, b)$ is the Inverse Gaussian distribution. The corresponding $\mathbb{G}$H-distribution is commonly named

$\mathbb{N}$ormal Inverse Gaussian

(notation $\mathbb{N} \circ \mathsf{IG}$).

(ii') $a > 0, b = 0, \nu > 0$: In this case $\mathbb{G}\mathsf{IG}(a, 0, \nu) = \mathsf{Gamma}(a/2, \nu)$ is the Gamma distribution. The corresponding $\mathbb{G}$H-distribution is named

$\mathbb{N}$ormal Gamma distribution

(notation : $\mathbb{N} \circ \mathsf{Gamma}$) or

$\mathbb{V}$G-distribution

(notation : $\mathbb{V}$G [Variance Gamma]).

(iii') $a > 0, b > 0, \nu = 1$: In this case $\mathbb{G}\mathsf{IG}(a, b, 1) = \mathsf{H}^+(a, b)$ is the Positive hyperbolic distribution. The corresponding $\mathbb{G}\mathsf{H}$-distribution is commonly named

Normal positive hyperbolic distribution

(notation : $\mathbb{H}$) or $\mathbb{N} \circ \mathsf{H}^+$.

Density, characteristic function, Lévy-measure can be simplified in the special cases.

Construction of Lévy processes

Having $\mathbb{G}$IG-distributions for σ^2 and $\mathbb{G}$H-distributions for h, we can turn to the construction of the Lévy process $H = (H_t)_{t \geq 0}$ used for modeling the prices $S_t = S_0 e^{H_t}$, $t \geq 0$.

There are two possibilities

— The fact that h is an infinitely divisible distribution allows one to construct, using the general theory, the Lévy process $H^* = (H^*_t)_{t \geq 0}$ such that

$$\mathrm{Law}(H^*_1) = \mathrm{Law}(h)\,.$$

— Using the constructed process $T = (T(t))_{t \geq 0}$, one forms the process $H = (H_t)_{t \geq 0}$:

$$H_t = \mu t + \beta T(t) + B_{T(t)}\,,$$

where the Brownian motion B and the process T are taken to be independent.

The processes $H = (H_t)_{t \geq 0}$ bears the name

L($\mathbb{G}$H) – Generalized hyperbolic Lévy processes

In the cases (i'), (ii'), and (iii') mentionned above the corresponding Lévy processes have the special names (i') $\mathsf{L}(\mathbb{N} \circ \mathsf{IG})$-process,

(ii') $\mathsf{L}(\mathbb{N} \circ \mathsf{Gamma})$- or $\mathsf{L}(\mathbb{V}\mathsf{G})$-process

(iii') $\mathsf{L}(\mathbb{N} \circ H^{+})$- or $\mathsf{L}(\mathsf{H})$-process.

It is interesting to mention that $\mathsf{L}(\mathbb{N} \circ \mathsf{IG})$- and $\mathsf{L}(\mathbb{N} \circ \mathsf{Gamma})$ have the important property that

$$\mathrm{Law}(H_t) \text{ belongs to the same type of distributions as } \mathrm{Law}(H_1).$$

Concluding remarks to $\mathsf{L}(\mathbb{G}\mathsf{H})$

— Densities of distributions of $h(= H_1)$ are determined by FIVE parameters (a, b, μ, β, ν), that gives a great freedom in determining parameters which would fit well the empirical data.

— The approach via independently time-changing Brownian motions has advantages related to simulation.

— In statistics there exist other methods in order to construct densities of distributions which would also fit well distributions of empirical data. The density $p^*(x; a, b, \mu, \beta, \nu)$ of $\mathbb{G}$H-distribution of (constructively built) variables $h = \mu + \beta\sigma^2 + \sigma\varepsilon$ has the essential advantage that $\mathbb{G}$H-distributions are

infinitely divisible

which enables us to construct processes $H = (H_t)_{t\geq 0}$ which describe adequately the time dynamics of logarithmic return of the prices $S = (S_t)_{t\geq 0}$.

Other popular Lévy models

Of course also other well-known Lévy processes are applied in mathematical finance. Among them

— The classical CGMY (Carr, Geman, Madan, Yor) model where we again have no centered gaussian term and the Lévy measure (describing the jump part(s)) has a density given by

$$f(x) = \frac{C}{|x|^{1+Y}} e^{-G|x|} 1\!\!1_{x<0} + \frac{C}{|x|^{1+Y}} e^{-M|x|} 1\!\!1_{x>0} \,,$$

with $C > 0$, $G > 0$, $M > 0$, $Y < 2$. For $Y = 0$ we obtain a different parametrization of a $\mathsf{L}(\mathbb{V}\mathsf{G})$-process.

— The Meixner process, again without centered gaussian term and the Lévy measure has again a density given by

$$f(x) = \delta \frac{\exp(\beta x/\alpha)}{x \sinh(\pi x/\alpha)}$$

where $\alpha > 0$, $-\pi < \beta < \pi$, $\delta > 0$.

To conclude it is important to stress that when working with exponential Lévy models, integrability of the asset price is not always guaranteed. A corresponding criteria for the existence of the first (and other) exponential moments can e.g. be derived from K. Sato, Lévy Processes and Infinitely Divisible Distributions (1999), Theorem 25.17.