Martingales in Financial Mathematics

Introduction to Financial Products

and

Martingales and the FTAP based on Finite Probability Spaces

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Outline of the Lecture Course

1. Martingales and the Fundamental Theorem(s) of Asset Pricing on finite probability spaces (including binomial and trinomial model).
2. The Snell envelope, optimal stopping, and American options on finite probability spaces (supermartingales and martingales).
3. Geometric Brownian motion and a martingale approach to the Black–Scholes formula (including exotic options).
4. Practical aspects of the Black–Scholes model.
5. On the theory of (no-)arbitrage in continuous time (local- and σ-martingales).
6. Selected topics on local- and stochastic volatility models ; Lévy driven models.

Perhaps(?) : Selected topics on interest rate modelling. New trends in financial mathematics, deep hedging.

References

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— (Shreve, S.E. (2004), Stochastic Calculus for Finance II, Springer Finance.)

Perhaps : Filipović, D. (2009), Term-Structure Models, Springer Finance ;

Platen, E., and Heath, D. (2006). A Benchmark Approach to Quantitative Finance, Springer Finance.

Outline of Chapter I

(1) Introduction

(2) Discrete time models

(a) Martingales and arbitrage opportunities

(b) Complete markets

(c) Cox, Ross and Rubinstein model

Introduction

Frequently considered questions in mathematical finance

— Investments in portfolios (optimisation)

— Valuation of financial assets and instruments (pricing)

— Risk management (e.g. hedging, solvency risks)

Selected historical remarks

— Bachelier (1901)

— Markowitz (1954)

— Black, Scholes and Merton (1973)

and

— Harrison and Pliska (1981)

— Dalang, Morton, Willinger (1990)

— Delbaen and Schachermayer (1998)

— Karatzas and Kardaras (2007) ; Platen and co-authors

Ongoing developments : Several new trends.

Mathematical modelling and analysis

— Probability theory (in particular stochastic calculus and martingale theory)

— Statistic

— Numerical analysis

— Machine learning

We will focuss on the first point.

One starting point: Balance sheet of a company

Assets	Liabilities and Owners' Equity
— current assets — liquid assets — accounts receivables — fixed assets — financial assets — tangible assets — other fixed assets	— liabilities — short-term liabilities — long-term liabilities (e.g. corporate bonds, . . .) — equity (e.g. common stock, . . .)

As an investor we can invest in bonds or in shares of the company.

As an investor we can invest in

— Bonds : Liabilities (for the company) $\Rightarrow$ the compensation is usually given by a fixed coupon (based on a fixed interest rate).

— Shares : Is a share of the equity $\Rightarrow$ the compensation is given by dividends (share of the profit).

— Hybrid instruments also exist.

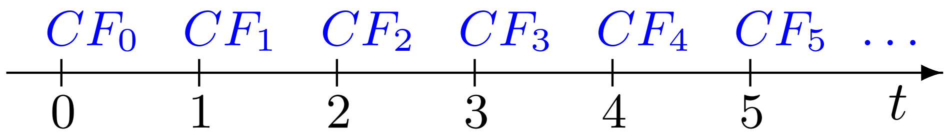
Note

— Fixed income instruments are also issued by the Federal Government, the so-called government bonds.

— Swiss government bonds can be considered as being risk-free (free of counterparty risk).

Valuation, a classical approach : Discount future cash flows

The Net Present Value (NPV) approach is rather popular



$$\text{NPV} = \sum_{t=0}^{N} \frac{CF_t}{(1 + i_t)^t} \, ,$$

— t date of the corresponding cash flow (CF),

— i_t interest rate, which reflects the cost of tying up capital / risks,

— CF_t net cash flow at t.

This approach is sometimes used for valuating bonds (but there exist also much more sophisticated approaches). It is often used for valuation of real estate and for the analysis of investment opportunities etc.

Financial derivatives

A *financial derivative* is an instrument whose value depends on, or is derived from, the value of another asset (underlying asset(s), sometimes simply called underlying(s) for short).

Examples for underlyings

- Shares
- Indices
- Currencies
- Interest rates
- Bonds
- Commodities
- Electricity
- Etc.

Share quotes etc. go up and down. . .

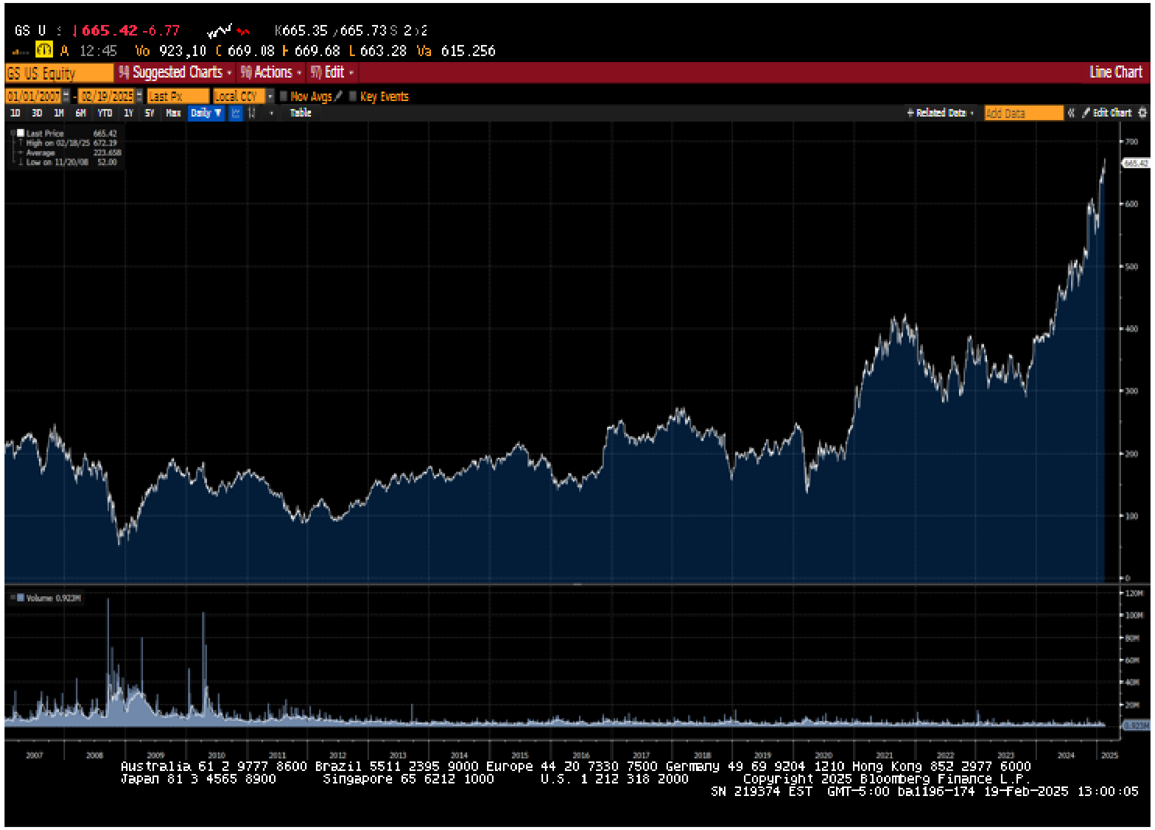


Figure 1 – Goldman Sachs-quotes (shares). *Source: Bloomberg*

Figure 2 – Credit Suisse. *Source: Bloomberg*

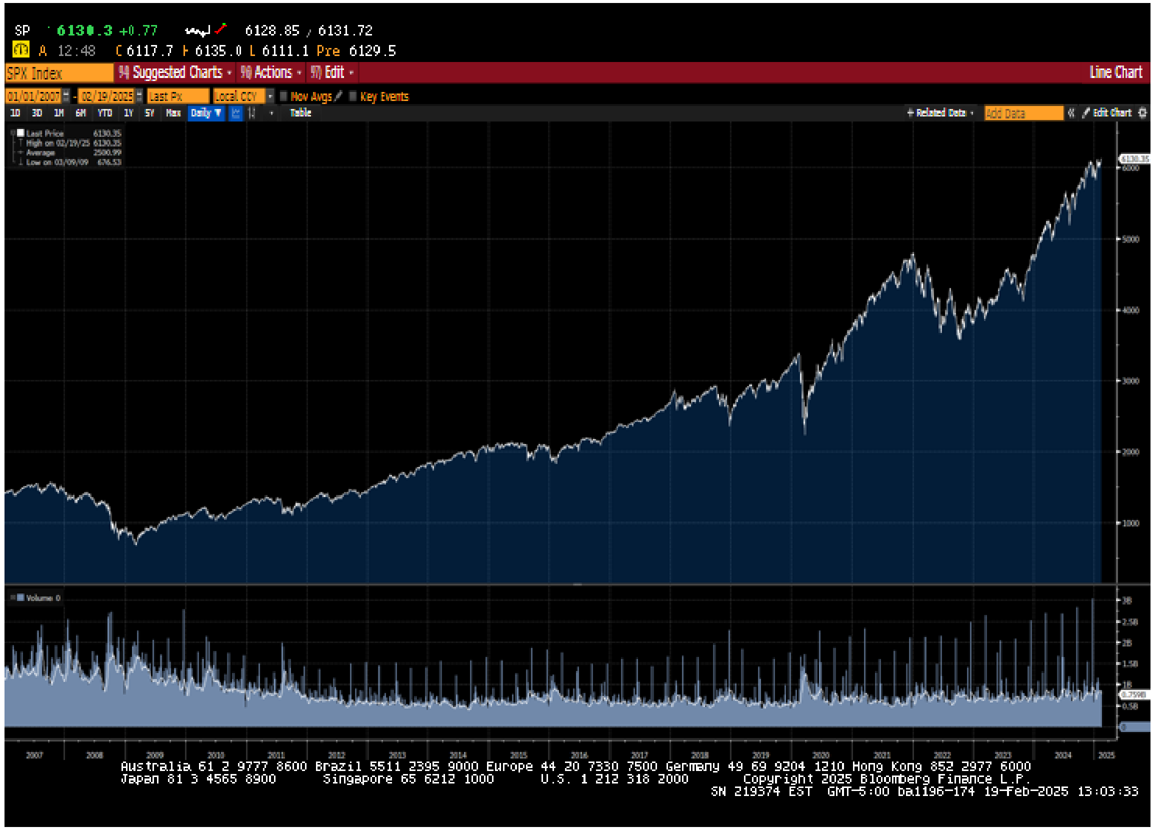


Figure 3 – S&P 500 index. *Source: Bloomberg*

Figure 4 – Euro quotes. *Source: Bloomberg*

Figure 5 – Swiss government bond. *Source: Bloomberg*

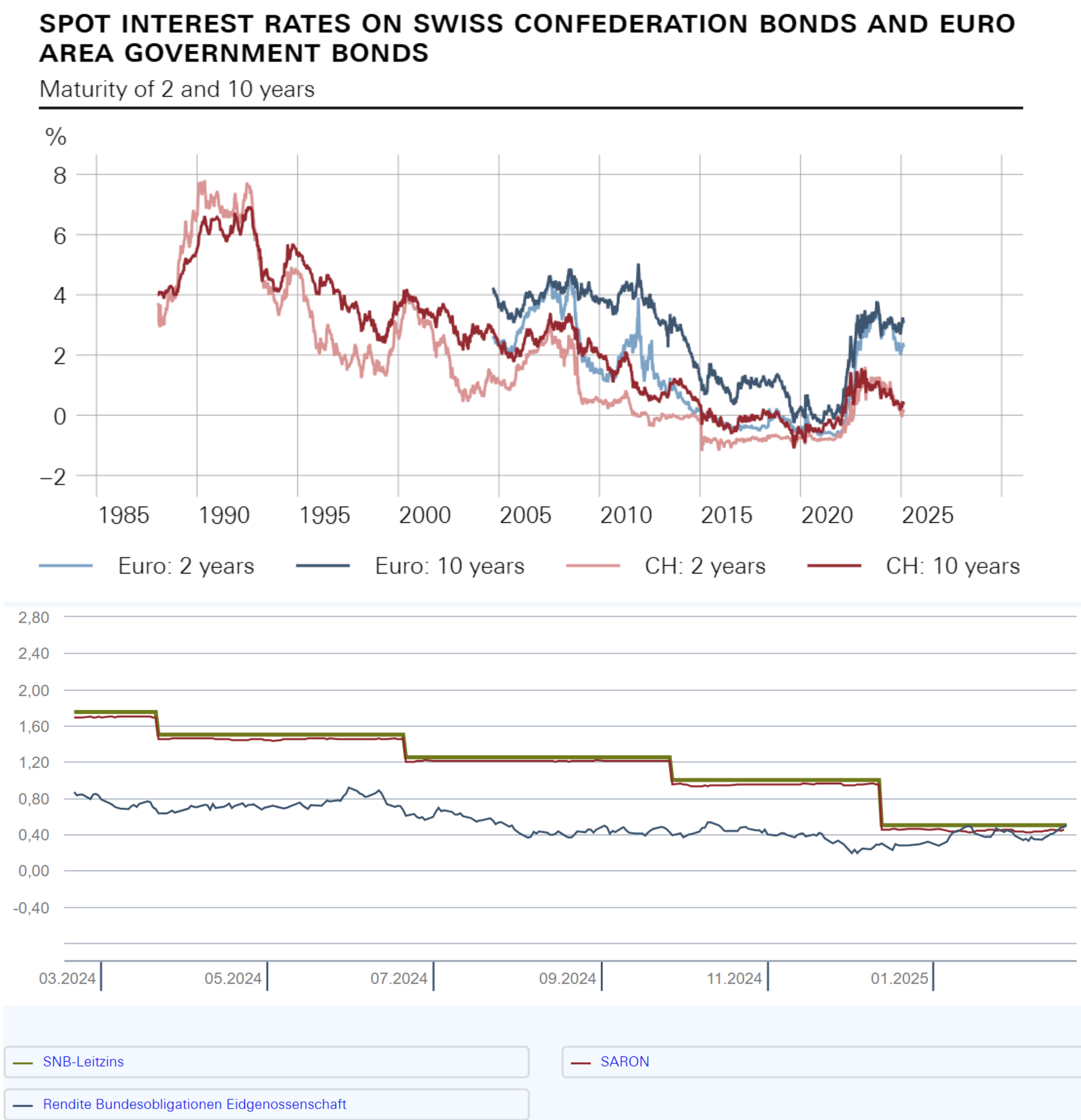


Figure 6 – Yields. *Source SNB*

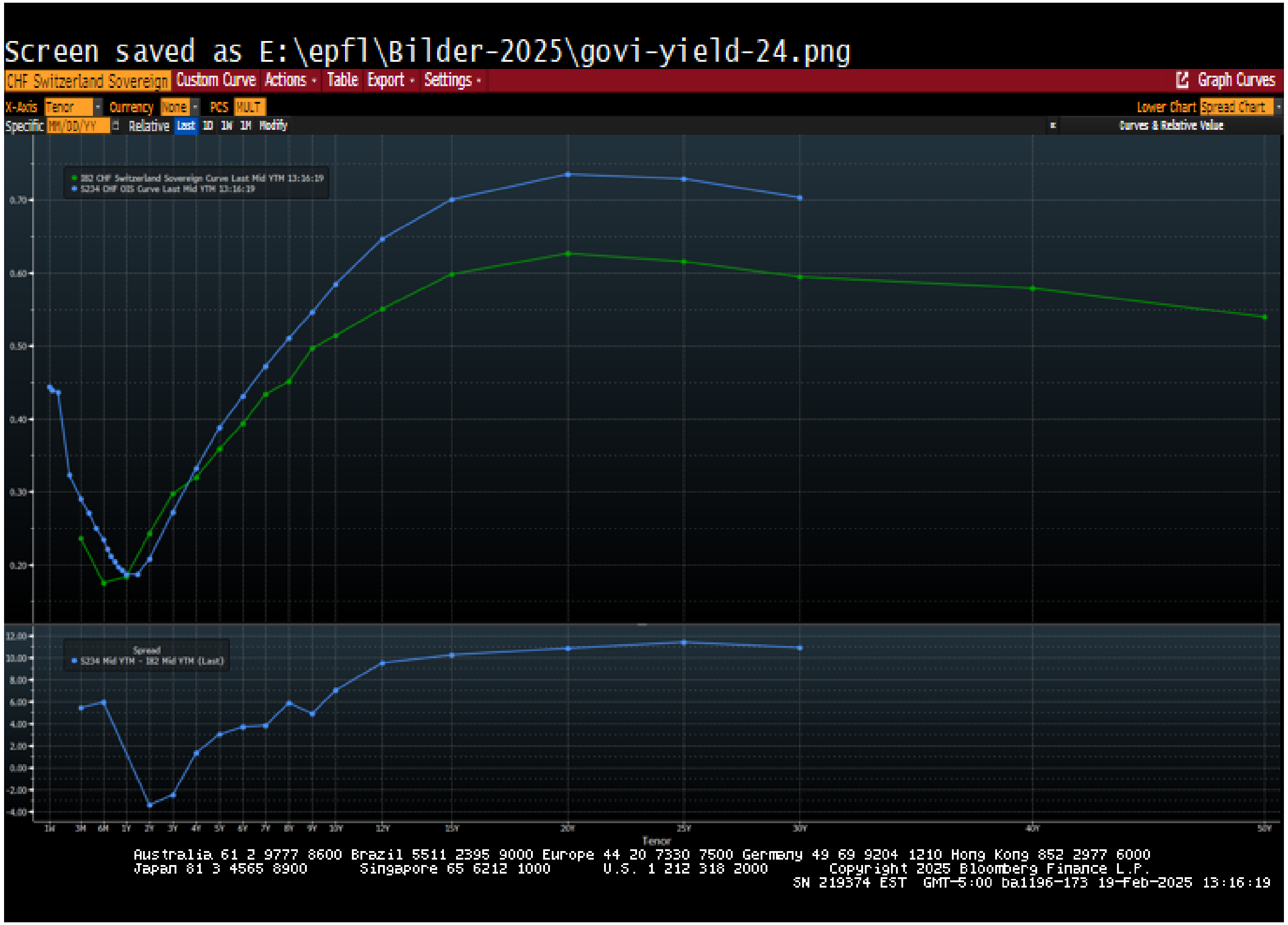
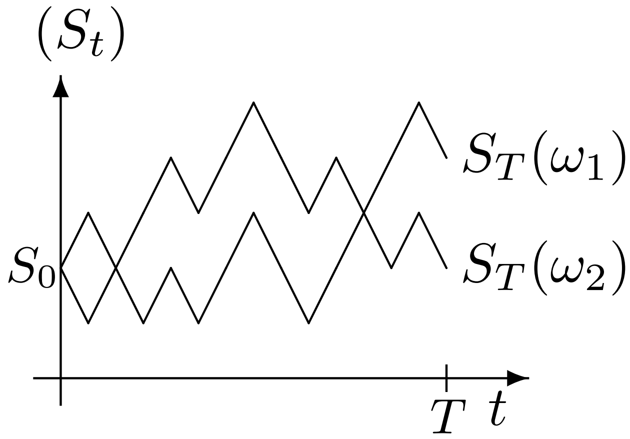


Figure 7 – Swiss Sovereign yield curve. *Source: Bloomberg*

Figure 8 – Swiss Sovereign yield curve (previous year). *Source: Bloomberg*

Mathematically speaking we can often define a derivative by a function f on a quote S_T in the future ($f : \mathbb{R}_+ \to \mathbb{R}$), or by a functional F on the price process.



Payoff of the derivative at maturity $T > 0$: $f(S_T)$ resp. $F((S_t)_{t\in[0,T]})$.

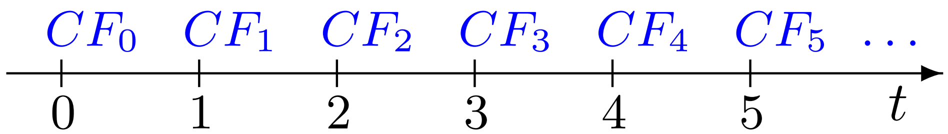
Typical examples

— Forward $f(S_T(\omega)) = S_T(\omega) - k$,

— European call option $f(S_T(\omega)) = \max(S_T(\omega) - k, 0)$, $k \geq 0$,

— European put option $f(S_T(\omega)) = \max(k - S_T(\omega), 0)$, $k \geq 0$.

Reminder : Discounting cash flows



$$\mathrm{NPV} = \sum_{t=0}^{N} \frac{CF_t}{(1+i_t)^t} \, ,$$

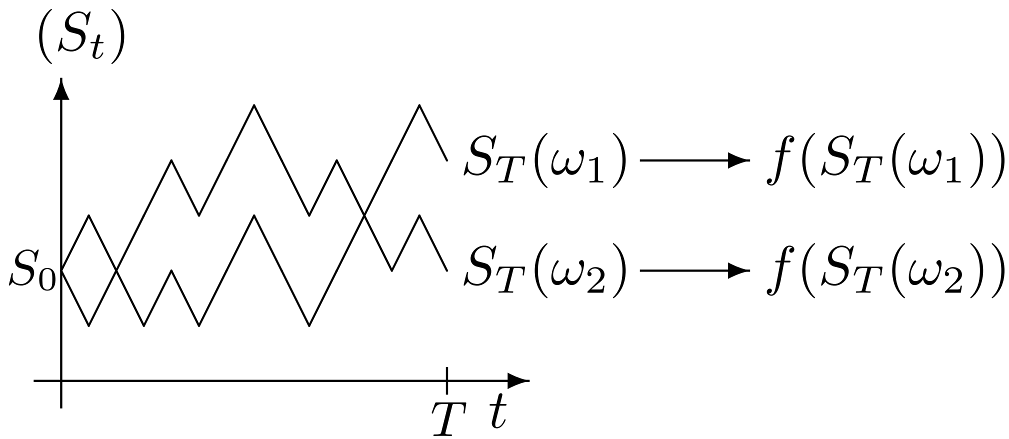
— t date of the corresponding cash flow (CF),

— i_t interest rate, which reflects the cost of tying up capital / risks,

— CF_t net cash flow at t.

Could / should we use this approach for deriving values of financial derivatives ?

Special structure of cash flows (very heuristic)



E.g. for payoffs of the form $f(S_T(\omega))$, i.e.

$$\Omega : \xrightarrow{S_T} \mathbb{R}_+ : \xrightarrow{f} \mathbb{R},$$

we have that $S_T(\omega)$ and $f(S_T(\omega))$ depend on the same $\omega \in \Omega$, where the payoff of the derivative is given by a deterministic function on $S_T \Rightarrow$ the price of the derivative and the price of the underlying asset should be in a certain relation.

Valuation approach for financial derivatives : Exclude arbitrage opportunities

Arbitrage opportunity : “To make money from nothing without risks”.

The idea for deriving a good price for a new product is :

Fix the price such that no arbitrage opportunity is created

(exclude arbitrage / no-arbitrage assumption (NA)).

Rough example : Forward price

The payoff at maturity $T > 0$ is given by

$$f(S_T) = S_T - k \,.$$

At $t = 0$ fix $k = F_{0,T}$ such that no premium is the “price” / payment at $t = 0$ (i.e. there are no cash flows at $t = 0$ / usual situation, also other situations are considered in the exercises).

Throughout this example we assume that the underlying asset is given by a non-dividend paying (until maturity $T > 0$) share. Furthermore, we assume that the risk-free interest rate (continuous compounding) r is constant.

Assume $F_{0,T} > S_0 e^{rT}$.

Then follow the following strategy:

At $t = 0$	CF
(1) Borrow money	S_0
(2) Buy spot the (cheap) share	$-S_0$
(3) Short the forward (short forward)	0
Total Cash flows at $t = 0$	0

At $t = T$	CF
Sell (2) for $F_{0,T}$ (based on (3))	$F_{0,T}$
Pay back the money (credit) incl. interest payment (1)	$-S_0 e^{rT}$
Total cash flow at $t = T$	$F_{0,T} - S_0 e^{rT}$

Since $F_{0,T} - S_0 e^{rT} > 0$ we have that this is an arbitrage strategy.

Assume that $F_{0,T} < S_0 e^{rT}$.

Then follow the following strategy:

At $t = 0$	CF
(1) Sell the (expensive) share (e.g. borrowed from a pension found)	S_0
(2) Pay the money into an account	$-S_0$
(3) Enter into a long position in the forward (long forward)	0
Total cash flow at $t = 0$	0

At $t = T$	CF
Close your bank account (2)	$S_0 e^{rT}$
Buy with the help of (3) a share for $F_{0,T}$ and settle (1)	$-F_{0,T}$
Total cash flow at $t = T$	$S_0 e^{rT} - F_{0,T}$

Since $S_0 e^{rT} - F_{0,T} > 0$ we observe that this is an arbitrage opportunity.

$$F_{0,T} < S_0 e^{rT} \quad \Rightarrow \quad \text{arbitrage} \quad \Leftarrow \quad F_{0,T} > S_0 e^{rT}$$

NA

$$\Downarrow$$

$$F_{0,T} = S_0 e^{rT} \,.$$

Rough remarks related to the forward example

If r is the expected yield of the share (continuous compounding) under $\mathbf{Q}$ (assumed to exist) and if we consider the price process as being a process indexed by $t \in \{0, T\}$, we have

(i) $\mathbf{E}_{\mathbf{Q}}[S_T] = S_0 e^{rT} \Rightarrow$ will not be the "real-world" measure (otherwise risk-neutral investors would be needed),

(ii) $e^{-rT}\mathbf{E}_{\mathbf{Q}}[S_T - F_{0,T}] = 0 \Rightarrow$ price to be paid at $t = 0$,

(iii) $\mathbf{E}_{\mathbf{Q}}[S_T/e^{rT}] = S_0 \Rightarrow$ the discounted price process is a martingale.

Note : Pt. (ii) corresponds to the NPV-philosophy, if we replace the CF by the expectation with respect to $\mathbf{Q}$ and if we discount with respect to r.

Translation of NA in the language of mathematics

There is a general translation of the economic assumption of NA in the language of mathematics (i.e. there is a result "behind" the observations on the last slide).

Very roughly speaking the *Fundamental Theorem of Asset Pricing* states that, essentially, a model of a financial market is free of arbitrage if and only if there is a probability measure $\mathbf{Q}$, equivalent to the original real-world measure $\mathbf{P}$ (i.e. $\mathbf{P}$ and $\mathbf{Q}$ vanish on the same events), such that the discounted asset price processes are martingales under $\mathbf{Q}$.

Measure $\mathbf{Q}$ is then called equivalent martingale measure.

In this case taking discounted expectations with respect to $\mathbf{Q}$ in order to price contingent claims yields *arbitrage-free pricing rules*, where $\mathbf{Q}$ runs through all equivalent martingale measures, i.e.

$$P_0^{\mathbf{Q}} = e^{-rT}\mathbf{E}_{\mathbf{Q}}[f(S_T)] \,, \quad \mathbf{Q}\text{–equivalent martingale measure.}$$

$\mathbf{Q}$ is often not unique !

This approach is often called risk-neutral valuation.

Models for the underlying assets

For pricing the forward in our example the behaviour (how the asset moves) of the price process until maturity was not important.

For more complex derivatives the situation is different, i.e. we need an asset price model for the underlying.

Geometric Brownian motion

The geometric Brownian motion (applied in the Black–Scholes model) is the probably most well-known asset price model.

$$S_t = S_0 \exp\left((\mu - \frac{1}{2}\sigma^2)t + \sigma W_t \right) \,, \quad \text{under } \mathbf{P} \,,$$

or in “local” form

$$dS_t = \mu S_t dt + \sigma S_t dW_t \,, \quad \text{under } \mathbf{P} \,,$$

where (W_t) stands for a Brownian motion under $\mathbf{P}$.

In particular we have that for a fixed t, W_t is normally distributed, with $\mathbf{E}[W_t] = 0$ and $\mathrm{Var}[W_t] = t$, so that

$$S_t = S_0 e^{\eta}\,,$$

where η is normally distributed with $\mathbf{E}[\eta] = (\mu - \frac{1}{2}\,\sigma^2)t$ and $\mathrm{Var}[\eta] = \sigma^2 t \Rightarrow$ $\mathbf{E}[S_t] = S_0 e^{\mu t} \Rightarrow \mu$ is the expected yield of the share (continuous compounding).

In the classical Black–Scholes setting the bond price process is of the form

$$B_t = e^{rt}.$$

In this market $\mathbf{Q}$ is unique $\Rightarrow$ prices of financial derivatives are also unique.

Furthermore, we will see that the asset price process under $\mathbf{Q}$ can be written as

$$S_t = S_0 \exp\left((r - \frac{1}{2}\,\sigma^2)t + \sigma \tilde{W}_t \right) \ , \quad \text{under } \mathbf{Q} \,,$$

or in “local” form

$$dS_t = r S_t dt + \sigma S_t d\tilde{W}_t \,, \quad \text{under } \mathbf{Q} \,,$$

where $(\tilde{W}_t)$ is a Brownian motion under $\mathbf{Q}$.

In view of that we often have

$$P_0 = e^{-rT}\mathbf{E}_{\mathbf{Q}}[f(S_T)] \,,$$

where $S_T = S_0 e^{\tilde{\eta}}$, for $\tilde{\eta}$ being normally distributed, with $\mathbf{E}_{\mathbf{Q}}[\tilde{\eta}] = (r - \frac{1}{2}\,\sigma^2)T$ and $\mathrm{Var}_{\mathbf{Q}}[\tilde{\eta}] = \sigma^2 T \Rightarrow \mathbf{E}_{\mathbf{Q}}[S_T] = S_0 e^{rT}$.

Example: European call option

$$e^{-rT}\mathbf{E}_{\mathbf{Q}}[(S_T - k)_+] = S_0\mathcal{N}(d_+) - ke^{-rT}\mathcal{N}(d_-) \,, \tag{1}$$

where $\mathcal{N}(\cdot)$ stands for the cumulative distribution function of a standard normally distributed random variable and

$$d_+ = \frac{\log\left(\frac{S_0}{k}\right) + \left(r + \frac{\sigma^2}{2}\right)\,T}{\sigma\sqrt{T}}\,, \quad d_- = \frac{\log\left(\frac{S_0}{k}\right) + \left(r - \frac{\sigma^2}{2}\right)\,T}{\sigma\sqrt{T}}\,.$$

We will derive the above pricing approach and the above formula with the help of martingale techniques.

Furthermore, we will briefly discuss the pros and cons of this model.

Some pros: Closed form formulas, hedging strategies can be derived efficiently, etc.

Cons: In particular the model cannot jointly explain traded option prices for options being written on one underlying.

If the formula would be correct, then the σ would be the same for all maturities and all strike prices.

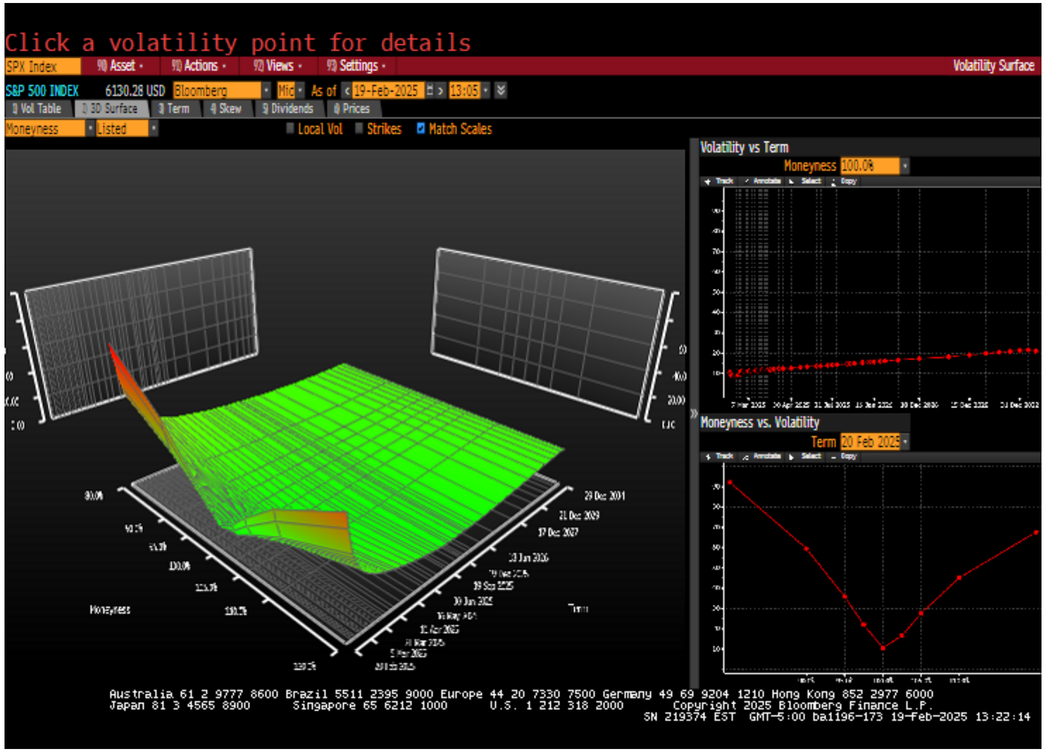


Figure 9 – *Source: Bloomberg*

After the crash 1987 people started to question the Black–Scholes model more intensively.

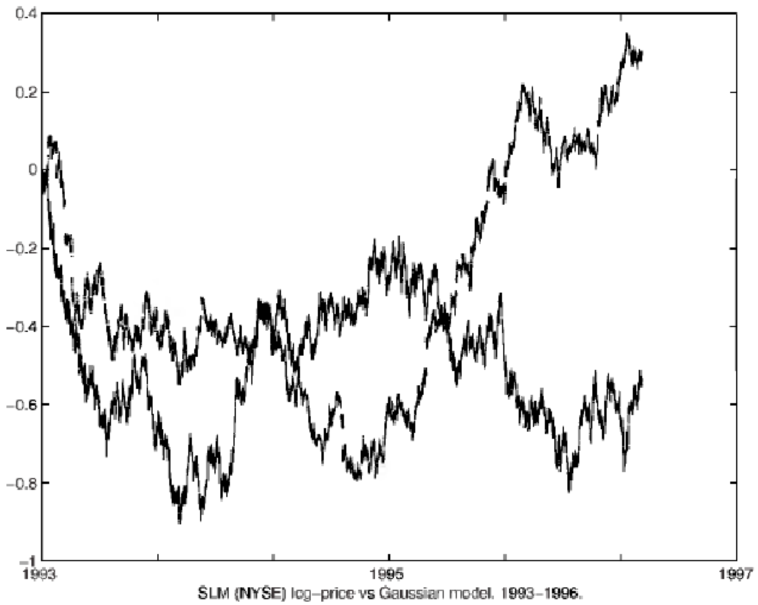


Figure 10 – *Source: Cont & Tankov, Financial Modelling with Jump Processes.*

$\Rightarrow$ We will discuss alternatives.

Short discussion of the NA-assumption

If I enter in a market with a new product it seems to be reasonable that I try to find a price, which excludes arbitrage opportunities for my counterparty.

However, are markets free of arbitrage ?

However, what are the consequences of absence of (sufficient) liquidity ?

Etc.

Perfect market hypothesis

Market structure characterized by a very large number of buyers and sellers of a homogeneous (nondifferentiated) product. Entry and exit from the industry is costless, or nearly so. Information is freely available to all market participants.

The following concrete simplifying assumptions are often used :

1. **No transaction costs or taxes, no bid-ask spread**.
2. No limitations on the quantities of transactions, securities are perfectly divisible and there are **no short-selling restrictions**.
3. **The market is liquid**.
4. *The risk-free lending and borrowing rate is the same*.

Discrete-time model

Discrete-time formalism

A discrete-time financial model is built on a finite probability space $(\Omega, \mathcal{F}, \mathbb{P})$ equipped with a filtration ($\{\mathcal{F}_n\}_{n=0,\,1,\,\ldots,\,N}$ with $N \in \mathbb{N}^*$), i.e. an increasing sequence of σ-algebras included in $\mathcal{F}$.

The horizon N will often correspond to the maturity of the considered options.

$\mathcal{F}_n$ can be interpreted as the information avilable at time n.

From now on, we assume

Standard filtration $\mathcal{F}_0 = \{\Omega, \emptyset\}$ and $\mathcal{F}_N = \mathcal{F}$.

Atoms $\mathbb{P}(\omega) > 0$, for all $\omega \in \Omega$.

Assets The market consists of $d+1$ assets with price at time n being given by $S_n = (S_n^0, S_n^1, \ldots, S_n^d)$.

Measurability The positive random variables S_n^0, S_n^1, …, S_n^d are measurable with respect to $\mathcal{F}_n$.

Risk-less asset There is a risk-less asset in the market such that $S_0^0 = 1$ and $S_n^0 = (1+r)^n$, with r being the positive risk-less interest rate.

Strategies

Definition 1 (Trading Strategy). *A trading strategy is defined as a stochastic process (i.e. a sequence in the discrete case)* $\phi = (\phi_n^0, \phi_n^1, \ldots, \phi_n^d)_{n=1,\ldots,N}$ *with values in* $\mathbb{R}^{d+1}$*, where* ϕ_n^i *denotes the number of shares of asset* i *held in the portfolio on the time interval between* $n-1$ *and* n*. The sequence* ϕ *is assumed to be predictable, i.e.* ϕ_j^i *is* $\mathcal{F}_{j-1}$*-measurable, for all* i *and* $j \geq 1$*,* ϕ_0^i *is* $\mathcal{F}_0$*-measurable for all* i*.*

The value $V_n(\phi)$ of the corresponding portfolio at time n is the scalar product

$$V_n(\phi) = \phi_n^0\, S_n^0 + \phi_n^1\, S_n^1 + \cdots + \phi_n^d\, S_n^d = \phi_n \cdot S_n\,.$$

Its discounted value is denoted by $\widetilde{V}_n(\phi) = V_n(\phi)\,(S_n^0)^{-1}$.
Analogously $\widetilde{S}_n = S_n\,(S_n^0)^{-1}$.

Self-financing strategies

A strategy is called **self-financing** if $\forall n \; \phi_{n+1} \cdot S_n = \phi_n \cdot S_n$.

The important interpretation is that at time n, when the new prices $(S_n^0, \ldots, S_n^d)$ are quoted, the investor readjust his position from ϕ_n to ϕ_{n+1} without bringing or consuming any wealth / money.

The profit or loss realised by following a self-financing strategy is only due to the price movements.

Proposition 1. *The following properties are equivalent*

I ϕ *is self-financing.*

II *For any* $n \in \{1, \ldots, N\}$,

$$V_n(\phi) = V_0(\phi) + \sum_{i=1}^{n} \phi_i \cdot \Delta S_i \,,$$

where $\Delta S_i = S_i - S_{i-1}$.

III *For any* $n \in \{1, \ldots, N\}$,

$$\widetilde{V}_n(\phi) = V_0(\phi) + \sum_{i=1}^{n} \phi_i \cdot \Delta \widetilde{S}_i \,,$$

where $\Delta \widetilde{S}_i = \widetilde{S}_i - \widetilde{S}_{i-1}$.

Note $\Delta \widetilde{S}^0_i = \widetilde{S}^0_i - \widetilde{S}^0_{i-1} = 0$.

Proposition 2. *For any predictable process* $(\phi_n^1, \ldots, \phi_n^d)_{n=0, \ldots, N}$ *and for any* V_0*, there exists a unique predictable process* $(\phi_n^0)_{n=0, \ldots, N}$ *such that the strategy* $\phi = (\phi_n^0, \phi_n^1, \ldots, \phi_n^d)$ *is self-financing and its initial value is* V_0*.*

Admissible strategies and arbitrage

Definition 2 (**Admissible strategy).** *A strategy ϕ is admissible if it is self-financing and if $V_n(\phi) \geq 0$ for any $n \in \{0, 1, \ldots, N\}$.*

Definition 3 (**Arbitrage).** *An arbitrage strategy is an admissible strategy with zero initial value and non-vanishing probability of an non-vanishing final value, i.e. $V_0 = 0$ and $\mathbb{P}\{V_N > 0\} > 0$.*

Martingales and arbitrage opportunities

Martingales (defined on a *finite* probability space)

Definition 4 (Martingales). *An adapted sequence* $\{M_n\}_{n=0,\,\ldots,\,N}$ *of real-valued random variables is*

Martingale *if* $M_n = \mathbb{E}[M_{n+1} \,|\, \mathcal{F}_n] \; \forall n$ *.*

Submartingale *if* $M_n \leq \mathbb{E}[M_{n+1} \,|\, \mathcal{F}_n] \; \forall n$*.*

Supermartingale *if* $M_n \geq \mathbb{E}[M_{n+1} \,|\, \mathcal{F}_n] \; \forall n$*.*

In particular

— If M_n is a martingale we have $\mathbb{E}[M_n] = \mathbb{E}[M_0]$.

— If M_n is a martingale and H_n is a predictable process then $X_n = \sum_{i=1}^{n} H_i \left(M_i - M_{i-1}\right)$ defines a martingale.

— The sum of two martingales is a martingale.

Proposition 3 (Characterisation of martingales). *An adapted sequence of real-valued random variables* (M_n) *is a martingale if and only if for any predictable sequence* (H_n)*, we have*

$$\mathbf{E}\left(\sum_{n=1}^{N} H_n \Delta M_n\right) = 0\,.$$

Viable financial markets

Definition 5 (**Viable market).** *A market is viable if there is no arbitrage opportunity.*

Theorem 1 (Fundamental Theorem of Asset Pricing)**.** *The market is viable if and only if there exists a probability measure* $\mathbb{Q}$ *equivalent to* $\mathbb{P}$ *such that the discounted prices of assets are* $\mathbb{Q}$*-martingales.*

Attention : The probability measure $\mathbb{Q}$ is not necessarily unique.

Complete markets and option pricing

A contingent claim is a nonnegative $\mathcal{F}_N$-measurable random variable.

Examples

European call $H = (S_N - K)^+$.

European put $H = (K - S_N)^+$.

European digital $H = \mathbb{I}_{S_N > K}$.

Definition 6 (**Attainable claim).** *The contingent claim defined by H is attainable if there exists an admissible strategy worth H at time N.*

Definition 7 (**Complete market).** *The market is complete if every contingent claim is attainable.*

Theorem 2 (**2. FTAP).** *A* viable *market is complete if and only if there exists a* **unique** *probability measure $\mathbb{Q}$ equivalent to $\mathbb{P}$, under which the discounted prices are martingales.*

Attention: The probability measure $\mathbb{Q}$ is a tool for deriving pricing formulas for options.

Pricing and hedging contingent claims in complete markets

Consider a viable and complete market with $\mathbb{Q}$ being the unique equivalent martingale measure.

Furthermore, consider a contingent claim H (a nonnegative $\mathcal{F}_N$-measurable random variable).

Under these assumptions there exists self-financing ϕ_H with $V_N(\phi_H) = H$.

We then have $\widetilde{V}_N(\phi_H) = \widetilde{H}$ and

$$\widetilde{V}_n(\phi_H) = \mathbb{E}^{\mathbb{Q}}\left[\widetilde{V}_N(\phi_H) \,|\, \mathcal{F}_n\right] = \mathbb{E}^{\mathbb{Q}}\left[\widetilde{H} \,|\, \mathcal{F}_n\right].$$

Since $H \geq 0 \Rightarrow V_n \geq 0$, hence, the strategy ϕ_H is also admissible.

Furthermore, at any time, the value of an admissible strategy replicating H is completely determined by $H \Rightarrow V_n(\phi_H)$ **is the value of the contingent claim at time** n.

If, at time 0, an investor sells the option for $\mathbb{E}^{\mathbb{Q}}[\widetilde{H}]$, he can follow a replicating strategy ϕ in order to generate an amount H at time $N \Rightarrow$ the investor is perfectly hedged.

It is important to notice that the computation of the option price only requires the knowledge of $\mathbb{Q}$ *and not of* $\mathbb{P}$*.*

Cox, Ross and Rubinstein model

Definition of the model

Trading dates $n = 0, 1, 2, \ldots, N$ with $N \in \mathbb{N}^*$

Risk-less asset $S_n^0 = (1+r)^n$ with $r > 0$

Risky asset there is only one risky asset with prices given by the vector

$(S_0, S_1, S_2, \ldots, S_N)$

Transition Assuming $-1 < a < b$

$$S_n = \begin{cases} S_{n-1}(1+a) \\ S_{n-1}(1+b) \end{cases}$$

$\forall n = 1, \ldots, N$

Probability space $\Omega = \{(1+a), (1+b)\}^N$, $\mathcal{F} = \mathcal{P}(\Omega)$, $\mathcal{F}_0 = \{\Omega, \emptyset\}$ and

$\mathcal{F}_n = \sigma(S_1, S_2, \ldots, S_n)$.

Some results

— It is necessary that $r \in (a,\ b)$ in order to end up with a viable market.

— The price of a European call is given by

$$C(0, S_0) = \frac{\sum_{j=0}^{N} \frac{N!}{(N-j)!\ j!} p^j\ (1-p)^{N-j}\ \left[S_0\,(1+a)^j\,(1+b)^{N-j} - K\right]^+}{(1+r)^N}.$$