Martingales in Financial Mathematics

Geometric Brownian motion

and the Black-Scholes model

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Chapter 3: GBM and the Black-Scholes model

— Probabilistic tools

— Pricing

— Hedging

Discrete time versus continuous time models

There are pros and cons of discrete time models relative to continuous time models, highly depending on, what you concretely want to do !

— Some computational advantages of discrete time models, e.g. related to American options.

— Nice closed form formulas based on some popular continuous time models for quite complicated derivatives, which are e.g. helpful for computing hedge parameters and for the speed (important on the "trading floor").

— Trading in reality is not continuous. However, how should we choose the minimal time tick ?

— Calibration tends to work better for continuous time models.

— Multivariate extensions are "easier" in continuous time models.

— Etc.

Historical remark on models for concrete dynamics

— Louis Bachelier “Théorie de la Speculation” A.S. ENS (1900)

— Fischer Black and Myron Scholes “The Pricing of Options and Corporate Liabilities”, J. of political Economy (1973)

— Hull & White (1987), Heston (1993), etc.: Stochastic volatilities

— Merton (76), Carr/Geman/Madan/Yor (2002/2003), Barndorff-Nielsen (. . .), Eberlein (. . .), etc., etc.: Models driven by Lévy processes (beyond Brownian motion driven models)

— Bates (1996): Stochastic volatilities with jumps

— . . .

Description of the model

The classical Black-Scholes model contains two assets

Risk free A risk-free asset. Its price process is denoted by $S^0 = (S^0_t)_{0\leq t\leq T}$. We assume that its dynamic is given by the ODE

$$dS^0_t = r\, S^0_t\, dt\,,$$

with $S^0_0 = 1$, where r is a non-negative constant. Obviously $S^0_t = e^{r\,t}$ (note that $r = r_c$).

Stock A risky asset with price process denoted by $S = (S_t)_{0\leq t\leq T}$. We assume that its dynamic is given by the SDE

$$dS_t = \mu\, S_t\, dt + \sigma\, S_t\, dB_t\,,$$

with $\sigma > 0$ and μ being constants and $(B_t)_{0\leq t\leq T}$ being a standard Brownian motion.

Filtration

As in the discrete time case we introduce a *filtration*. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. Then a filtration $(\mathcal{F}_t)$ is an increasing family of σ-subalgebras of $\mathcal{F}$. We can construct a filtration based on a process (X_t) by $\bar{\mathcal{F}}_t = \sigma(X_s, s \leq t)$.

However, from now on, we assume that the filtration satisfies the *usual conditions*, i.e.

— If $A' \subset A \in \mathcal{F}$ and if $\mathbb{P}(A) = 0$, then for any t, $A' \in \mathcal{F}_t$.

— $\mathcal{F}_t = \bigcap_{s>t} \mathcal{F}_s$.

E.g. if (X_t) is a Brownian Motion and if $\mathcal{F}_t$ is the σ-algebra generated by both, $\bar{\mathcal{F}}_t$ and $\mathcal{N}$ (the σ-algebra generated by all the $\mathbf{P}$-null sets of $\mathcal{F}$), we obtain a filtration satisfying the usual conditions (called the standard Brownian filtration).

Standard model for the price

The unique (for given S_0) solution (S_t) of the SDE of the risky asset is given by

$$S_t = S_0 \, e^{\sigma B_t + (\mu - \frac{\sigma^2}{2}) \, t} \, ,$$

where S_0 stands for the price at $t = 0$.

Properties of the risky asset

continuous pathes

independence of the *relative* increments If $s \leq t$, $\dfrac{S_t}{S_s}$ and $\dfrac{S_t - S_s}{S_s}$ are independent of $\mathcal{F}_s$.

stationarity of the relative increments if $u \leq t$ the law of $\dfrac{S_t - S_u}{S_u}$ shares the distribution with $\dfrac{S_{t-u} - S_0}{S_0}$.

log normality $\forall t$, we have that $\log S_t \sim \mathcal{N}\left(\log S_0 + \left(\mu - \dfrac{\sigma^2}{2}\right)\, t,\, \sigma^2\, t\right)$.

Strategies

Definition 1 (Strategy)**.** *A strategy is a process* $\phi = (\phi_t)_{0 \le t \le T} = (H_t^0\,, H_t)_{0 \le t \le T}$ *with values in* $\mathbb{R}^2$*, adapted to the filtration* $(\mathcal{F}_t)_{0 \le t \le T}$*.*

The components H_t^0 and H_t are the quantities of the risk-less asset and the risky asset, respectively.

The value of the portfolio at time t is given by

$$V_t(\phi) = H_t^0\, S_t^0 + H_t\, S_t\,.$$

Self-financing strategies

Definition 2 (Self-financing strategy)**.** *A strategy ϕ is self-financing if it satisfies the following two conditions*

Integrability $\int_0^T |H_t^0|\, dt + \int_0^T H_t^2\, dt < \infty\, a.s.$

Self-financing $dV_t = d(H_t^0\, S_t^0) + d(H_t\, S_t) = H_t^0\, dS_t^0 + H_t\, dS_t$, *i.e.*

$$V_t(\phi) = H_t^0\, S_t^0 + H_t\, S_t = H_0^0\, S_0^0 + H_0\, S_0 + \int_0^t H_s^0\, dS_s^0 + \int_0^t H_s\, dS_s\,,$$

a.s., for all $t \in [0, T]$.

Denote the present values of the risky asset and the portfolio, respectively, by $\widetilde{S}_t = e^{-r\,t}\, S_t$ and $\widetilde{V}_t = e^{-r\,t}\, V_t$.

Proposition 1. *Let* $\phi = (\phi_t)_{0\le t\le T} = (H_t^0\,, H_t)_{0\le t\le T}$ *be a strategy such that* $\int_0^T H_t^2\, dt + \int_0^T |H_t^0|\, dt < \infty$ *a.s. Then* ϕ *defines a self-financing strategy if and only if*

$$\widetilde{V}_t(\phi) = V_0(\phi) + \int_0^t H_s\, d\widetilde{S}_s \quad \textit{a.s.}$$

for all $t \in [0, T]$.

Equivalent probability measures

Definition 3 (Absolute continuity)**.** *A probability measure* $\mathbb{Q}$ *on* $(\Omega, \mathcal{F})$ *is absolutely continuous with respect to* $\mathbb{P}$ *if* $\forall A \in \mathcal{F}$, $\mathbb{P}(A) = 0 \Rightarrow \mathbb{Q}(A) = 0$, *in which case we write* $\mathbb{Q} \ll \mathbb{P}$.

Two probability measures $\mathbb{P}$ and $\mathbb{Q}$ are equivalent if $\mathbb{P} \ll \mathbb{Q}$ and $\mathbb{Q} \ll \mathbb{P}$.

Theorem 1 (Radon-Nikodym)**.** *A probability measure* $\mathbb{Q}$ *is absolutely continuous with respect to* $\mathbb{P}$ *if and only if there exists a non-negative random variable* Z *on* $(\Omega, \mathcal{F})$ *such that* $\forall A \in \mathcal{F}$

$$\mathbb{Q}(A) = \int_A Z \, d\mathbb{P} \, .$$

The random variable Z *is the density of* $\mathbb{Q}$ *with respect to* $\mathbb{P}$ *and denoted by* $\dfrac{d\mathbb{Q}}{d\mathbb{P}}$.

The Girsanov theorem

Theorem 2 (Girsanov)**.** *Let $\theta = (\theta_t)_{t\in[0,T]}$ be an adapted process satisfying $\int_0^T \theta_s^2\, ds < \infty$ a.s. and such that the process $(L_t)_{t\in[0,T]}$ defined by*

$$L_t = e^{-\int_0^t \theta_s\, dB_s - \frac{1}{2}\int_0^t \theta_s^2\, ds}$$

is a martingale. Then under the probability $\mathbb{P}^{(L)}$ with density L_T with respect to $\mathbb{P}$, the process $(W_t)_{t\in[0,T]}$ defined by

$$W_t = B_t + \int_0^t \theta_s ds\,,$$

is an $(\mathcal{F}_t)$-standard Brownian motion.

Here we have $\frac{d\mathbb{P}^{(L)}}{d\mathbb{P}}|_{\mathcal{F}_t} = \mathbf{E}[L_T \,|\, \mathcal{F}_t] = L_t$.

Representation of Brownian martingales

Theorem 3 (Martingale Representation Theorem)**.** *Let* $M = (M_t)_{0 \leq t \leq T}$ *be a square-integrable martingale, with respect to the filtration* $(\mathcal{F}_t)_{0 \leq t \leq T}$ generated by standard Brownian motion $(W_t)_{0 \leq t \leq T}$*. There exits an adapted process* $(K_t)_{0 \leq t \leq T}$ *such that* $\mathbb{E}[\int_0^T K_u^2 \, du] < \infty$ *and*

$$\forall t \in [0, T] \quad M_t = M_0 + \int_0^t K_u \, dW_u \quad \textit{a.s.}$$

A probability measure under which $(\tilde{S}_t)$ is a martingale

In view of the SDE for (S_t) we have

$$\begin{aligned} d\widetilde{S}_t &= -re^{-rt}S_t\,dt + e^{-rt}\,dS_t \\ &= \tilde{S}_t((\mu - r)dt + \sigma\,dB_t) \end{aligned}$$

Hence, if we set $W_t = B_t + \frac{\mu - r}{\sigma}t$, we have

$$d\tilde{S}_t = \tilde{S}_t \sigma dW_t\,.$$

Theorem 2, for $\theta_t = \frac{\mu - r}{\sigma}$, implies the existence of a probability measure $\mathbb{Q}$ equivalent to $\mathbb{P}$ under which $(W_t)_{t\in[0,T]}$ is a standard Brownian motion.

Then for the discounted asset price process $(\tilde{S}_t)$ we have

$$\tilde{S}_t = S_0 \exp(\sigma W_t - \sigma^2 t/2)$$

under $\mathbb{Q}$. Hence, $(\tilde{S}_t)$ is a $\mathbb{Q}$-martingale.

Pricing

Definition 4 (Admissible strategy)**.** *A strategy* $\phi = (H_t^0\,, H_t)_{0\leq t\leq T}$ *is admissible if it is self-financing and if the discounted value* $\widetilde{V}_t(\phi) = H_t^0 + H_t\widetilde{S}_t$ *of the corresponding portfolio is, for all* t*, non-negative, and if* $\mathbb{E}_{\mathbb{Q}}[\int_0^T H_t^2(\sigma\widetilde{S}_t)^2\,dt] < \infty$*.*

Theorem 4 (Pricing contingent claims)**.** *In the Black–Scholes model, any option defined by a non-negative* $\mathcal{F}_T$*-measurable random variable* h*, which is square-integrable under the probability measure* $\mathbb{Q}$*, is replicable (attainable) and the value at time* t *of any replicating portfolio is given by*

$$V_t = e^{-r\,(T-t)}\mathbb{E}^{\mathbb{Q}}[h\,|\,\mathcal{F}_t]\,.$$

Hence, the option value at time t *can be naturally defined by the expression* $e^{-r\,(T-t)}\mathbb{E}^{\mathbb{Q}}[h\,|\,\mathcal{F}_t]\,.$

European call option

In the Black–Scholes model the price of an European call with maturity $T > 0$ and strike $K > 0$ is given by

$$C_t = S_t \mathcal{N}(d_+) - K\, e^{-r(T-t)}\, \mathcal{N}(d_-)\,,$$

where

$\mathcal{N}$ is the cdf of a standard normally distributed random variable and

$d_\pm$ are given by $d_\pm = \dfrac{\log\left(\frac{S_t}{K}\right) + \left(r \pm \frac{\sigma^2}{2}\right)\,(T-t)}{\sigma\sqrt{T-t}}$.

Furthermore, the hedging strategy is given by

$$\phi_t = (H_t^0,\, H_t) = \left(-K\, e^{-rT}\, \mathcal{N}(d_-),\, \mathcal{N}(d_+)\right).$$

Hedging

In cases where the claim is defined by a random variable of the form $h = f(S_T)$, it is often possible to derive the replication portfolio explicitly. A replication portfolio must have, at any time t, a discounted value equal to

$$\widetilde{V}_t = e^{-rt} F(t, S_t)\,,$$

where F is defined by

$$F(t,x) = e^{-r(T-t)} \int_{\mathbb{R}} f\big(x e^{(r-\frac{1}{2}\,\sigma^2)(T-t)+\sigma\sqrt{T-t}z}\big) \frac{e^{-\frac{z^2}{2}}}{\sqrt{2\pi}}\, dz\,.$$

Hence, F will usually be quite regular. If we set

$$\tilde{F}(t,x) = e^{-rt} F(t, x e^{rt})\,,$$

we have $\widetilde{V}_t = \widetilde{F}(t, \widetilde{S}_t)$, and for all $t < T$, we obtain from the Itô Lemma

$$\widetilde{F}(t, \widetilde{S}_t) = \widetilde{F}(0, S_0) + \int_0^t \underbrace{\left(\frac{\partial \widetilde{F}}{\partial t}(u, \widetilde{S}_u) + \frac{1}{2}\,\sigma^2 \widetilde{S}_u^2 \frac{\partial^2 \widetilde{F}}{\partial x^2}(u, \widetilde{S}_u)\right)}_{K_u} du$$

$$+ \int_0^t \sigma \widetilde{S}_u \frac{\partial \widetilde{F}}{\partial x}(u, \widetilde{S}_u) dW_u \,.$$

The martingale property of $(\widetilde{F}(t, \widetilde{S}_t))$ under $\mathbb{Q}$ implies that K_u is vanishing. Hence,

$$\widetilde{F}(t, \widetilde{S}_t) = \widetilde{F}(0, S_0) + \int_0^t \frac{\partial \widetilde{F}}{\partial x}(u, \widetilde{S}_u) d\widetilde{S}_u \,.$$

Thus,

$$H_t = \frac{\partial \widetilde{F}}{\partial x}(t, \widetilde{S}_t) = \frac{\partial F}{\partial x}(t, S_t)\,.$$

If we set $H_t^0 = \widetilde{F}(t, \widetilde{S}_t) - H_t \widetilde{S}_t$, we have that $\left(H_t^0, H_t\right)$ is self-financing and that $V_t = F(t, S_t)$, being non-negative for $\mathbb{R}_+$-valued payoff functions f.