Martingales in Financial Mathematics: Static and semi-static hedging

Week 9, 2025

We assume a risk-neutral Black–Scholes setting, i.e. we have a risk-less bond with price process $B_t = e^{rt}$, $t \in [0,T]$, $r > 0$ and a risky asset with a price process satisfying

$$dS_t = rS_t dt + \sigma S_t dW_t \,,$$

where $\sigma > 0$ and $(W_t)_{t\in[0,T]}$ is a standard Brownian motion (with respect to $\mathbb{Q}$).

Exercise 1: Forward price

Recall that a forward contract is an agreement to pay a specified delivery price k at a maturity date $T \geq 0$ for the asset whose price at time t is S_t.

The T-forward price $F_{t,T}$ of this asset at time $t \in [0,T]$ is the value of k that makes the forward contract have no-arbitrage price zero at time t. Describe the process $F_{t,T}$, $t \in [0,T]$.

Exercise 2: European put-call symmetry

Show that in the Black–Scholes case the European put-call symmetry holds, which can be expressed by the property that for arbitrary $k \geq 0$

$$\begin{aligned} e^{-r(T-t)}\mathbb{E}_{\mathbb{Q}}[(F_{t,T}\eta_{t,T} - k)_+|\mathcal{F}_t] &= e^{-r(T-t)}\mathbb{E}_{\mathbb{Q}}[(F_{t,T} - k\eta_{t,T})_+|\mathcal{F}_t] \\ &= e^{-r(T-t)}\frac{k}{F_{t,T}}\mathbb{E}_{\mathbb{Q}}\Big[\big(\frac{F_{t,T}^2}{k} - F_{t,T}\eta_{t,T}\big)_+|\mathcal{F}_t\Big] \,, \end{aligned}$$

where $\eta_{t,T} = e^{-\frac{1}{2}\sigma^2(T-t)+\sigma(W_T-W_t)}$, $t \in [0,T]$, so that $F_{t,T}\eta_t = S_T$ under $\mathbb{Q}$, i.e. we obtain a relation between European call and put prices (often this relation is only formulated for $t = 0$, i.e. for the random variable S_T).
Hint: Write the Balck–Scholes formulas in terms of $F_{t,T}$.

Exercise 3: Semi-static hedge of a down-and-out call

Semi-static hedging strategies are often defined to be replicating strategies where trading is no more needed than two times after inception. Assume that there is a barrier H, a strike k satisfying $H < k$, where $S_0 > H$, and assume that there are no carrying costs. The down-and-out call is knocked-out if H is hit any time before maturity. Otherwise pays $(S_T - k)_+$, i.e.

$$X_{\text{doc}} = (S_T - k)_+ \mathbb{1}_{S_t > H \,, \forall t\in[0,T]} \,.$$

Assume that European calls and puts are available for arbitrary strikes. Use the European put-call symmetry in order to derive a semi-static hedging strategy (use without proof that the put-call symmetry from Exercise 2 also holds for $[0,T]$-valued stopping times).

Exercice 4: Decomposition of European payoff functions

Assume that a payoff function $f\colon \mathbb{R}_+ \to \mathbb{R}$ is two times *continuously* differentiable. Show that for $a \in \mathbb{R}_+$

$$f(x) = f(a) + f'(a)(x-a) + \int_a^\infty f''(k)(x-k)_+ dk + \int_0^a f''(k)(k-x)_+ dk\,,$$

and give an economic interpretation.

Exercise 5: Implicit distribution

Consider a risk-neutral setting where $\tilde{S}_T$ is sampled from a martingale and assume for simplicity that the strictly positive random variable S_T is absolutely continuous with continuous density q. Show that either the prices of European calls or European puts for arbitrary strikes uniquely determine the distribution of S_T.[1]

[1] This result is obviously interesting in its own right but has also some consequences for the caracterisation of random variables satisfying European put-call symmetry.