Exercises Martingales in Financial Mathematics: Black–Scholes and Brownian Motion

Week 8, 2025

Recall that in a Black–Scholes setting, the price of a European call is given by

$$V_t = C(T-t,\, S_t,\, K,\, r,\, \sigma) = S_t\,\mathcal{N}(d_+) - K\,e^{-r(T-t)}\,\mathcal{N}(d_-)$$

with

$$d_\pm = \frac{\log\frac{S_t}{K} + \left(r \pm \frac{1}{2}\sigma^2\right)\,(T-t)}{\sigma\sqrt{T-t}}\,.$$

Exercise 1: Black–Scholes model and Greeks

We are interested in the prices of European call and put options. In particular in the partial derivatives of the pricing function with respect to model parameters.

1. Use the European put-call parity in order to (re)derive the pricing formula for European puts.
2. Compute the deltas

$$\begin{aligned}\Delta_C(T-t,\, S_t,\, K,\, r,\, \sigma) &= \frac{\partial C}{\partial x}(T-t,\, x,\, K,\, r,\, \sigma)\Big|_{x=S_t}\\ \Delta_P(T-t,\, S_t,\, K,\, r,\, \sigma) &= \frac{\partial P}{\partial x}(T-t,\, x,\, K,\, r,\, \sigma)\Big|_{x=S_t}\end{aligned}$$

3. Compute the gammas

$$\begin{aligned}\Gamma_C(T-t,\, S_t,\, K,\, r,\, \sigma) &= \frac{\partial^2 C}{\partial x^2}(T-t,\, x,\, K,\, r,\, \sigma)\Big|_{x=S_t}\\ \Gamma_P(T-t,\, S_t,\, K,\, r,\, \sigma) &= \frac{\partial P^2}{\partial x^2}(T-t,\, x,\, K,\, r,\, \sigma)\Big|_{x=S_t}\end{aligned}$$

4. Compute the vegas

$$\begin{aligned}\mathrm{Vega}_C(T-t,\, S_t,\, K,\, r,\, \sigma) &= \frac{\partial C}{\partial x}(T-t,\, S_t,\, K,\, r,\, x)\Big|_{x=\sigma}\\ \mathrm{Vega}_P(T-t,\, S_t,\, K,\, r,\, \sigma) &= \frac{\partial P}{\partial x}(T-t,\, S_t,\, K,\, r,\, x)\Big|_{x=\sigma}\end{aligned}$$

5. Compute the thetas

$$\Theta_C(T-t,\, S_t,\, K,\, r,\, \sigma) \;=\; -\frac{\partial C}{\partial s}(s,\, S_t,\, K,\, r,\, \sigma)\bigg|_{s=T-t}$$

$$\Theta_P(T-t,\, S_t,\, K,\, r,\, \sigma) \;=\; -\frac{\partial P}{\partial s}(s,\, S_t,\, K,\, r,\, \sigma)\bigg|_{s=T-t}$$

6. Compute the rhos

$$\rho_C(T-t,\, S_t,\, K,\, r,\, \sigma) \;=\; \frac{\partial C}{\partial p}(T-t,\, S_t,\, K,\, p,\, \sigma)\bigg|_{p=r}$$

$$\rho_P(T-t,\, S_t,\, K,\, r,\, \sigma) \;=\; \frac{\partial P}{\partial p}(T-t,\, S_t,\, K,\, p,\, \sigma)\bigg|_{p=r}$$

7. Give some remarks on Δ, Γ, Vega, and Θ.

Exercise 2: An numerical example of an application of the Black–Scholes model

A stock price follows geometric Brownian motion with an expected return of 16% and a volatility of 35%. The current price is \$38.
(a) What is the probability that a European call option with strike price of \$40 maturing in three months will be exercised? What is the value of this option at maturity (assume 10%p.a. risk-free interest rate)?
(b) Answer the same questions for a European put option?

Exercise 3: Black–Scholes Model: Another financial derivative

Compute the price and describe the replication strategy at $t = 0$ of the European derivative being defined by the following payoff function (where we assume that the asset price process follows a geometric Brownian motion and where $k > 0$ is a positive constant).

$$f(S_T) = \max(S_T, k)\,.$$

Exercise 4: Brownian motion

Let (W_t) be a standard Brownian motion. Which one of the following processes are standard Brownian motions as well (justify your answers)?

1. The process (X_t), being defined by $X_t = 2(W_{1+\frac{t}{4}} - W_1)$.

2. The process (Y_t), being defined by $Y_t = \sqrt{t}W_1$.

3. The process (Z_t), being defined by $Z_t = W_{2t} - W_t$.