# Exercises Martingales in Financial Mathematics: Aspects of Brownian Motion and Barrier Options

Week 7, 2025

## Exercise 1: Reflection principle

Let $(W_t)$ be a standard Brownian motion and $(\mathcal{F}_t)$ the corresponding Brownian filtration (as introduced in the lecture course). Let $M_t$ be the corresponding running maximum, i.e. $M_t = \sup_{s \le t} W_s$. Derive by a heuristic argument or by a proof that for $w \le m$, $m > 0$

$$\mathbb{P}(M_t \ge m, W_t \le w) = \mathbb{P}(W_t \ge 2m - w)\,.$$[1]

## Exercise 2: Joint distribution of $W_t$ and $M_t$

For a $t > 0$, find the joint probability density function of $M_t = \sup_{s \in [0,t]} W_s$ and $W_t$ for $w \le m$, $m > 0$.
Hint: Use Exercise 1.

## Exercise 3: Brownian motion with drift

Let $(W_t)_{t \in [0,T]}$ be a standard Brownian motion defined on $(\Omega\,, \mathcal{F}, \mathbb{Q})$ and let $\hat{W} = (\alpha t + W_t)_{t \in [0,T]}$ for a given real $\alpha$, i.e. the Brownian motion $(\hat{W}_t)$ has drift $\alpha$ under $\mathbb{Q}$. We further define $\hat{M}_T = \sup_{0 \le t \le T} \hat{W}_t$. Show that for $m > 0$, $w \le m$, the joint density function of $(\hat{M}_T, \hat{W}_T)$ under $\mathbb{Q}$ is given by

$$\tilde{f}(m, w) = \frac{2(2m - w)}{T\sqrt{2\pi T}} e^{\alpha w - \frac{1}{2}\alpha^2 T - \frac{1}{2T}(2m-w)^2}\,. \tag{1}$$

Hint: Use Girsanov for the density $\hat{Z}_t = e^{-\alpha W_t - \frac{1}{2}\alpha^2 t}$ and the result from Exercise 2.

[1] Hints for a proof: For a proof use the fact that for a stopping time $\tau$ with $\mathbb{P}(\tau < \infty) > 0$ we have that conditionally on $\{\tau < \infty\}$ the process $(W_{t+\tau} - W_\tau, t \ge 0)$ is a $(\mathcal{F}_{\tau+t})$-Brownian motion independent of $\mathcal{F}_\tau$ and that for positive $m$ $S_m := \inf\{t : W_t \ge m\} = \inf\{t : W_t = m\} =: T_m$ is/are a.s. finite stopping time(s) (you are not expected to proof that, the interested student is referred e.g. to Revuz and Yor, Continuous Martingales and Brownain Motion, Sec. 3, Ch. II and Sec. 3, Ch. III; or for a rather elementary proof of the strong Markov Property to Th. 32 in P. E. Protter, Stochastic Integral and Differential Equations (Version 2.1) and for $T_m$ to be a stopping time to Th. 4 in the same book).

## Exercise 4: Value of a up-and-out call

With the usual notation, price (at $t = 0$) the following so-called up-and-out call being defined by

$$h = (S_T - k)_+ \mathbb{1}_{S_t < b, \forall t \in [0,T]}$$

where we assume that $S_0 < b$ and $0 < k < b$ (otherwise, the option must be knocked out in order to be in the money and hence, could only pay off zero).