Exercises Martingales in Financial Mathematics: Borwnian motion driven models

Week 6, 2025

Exercise 1: Black-Scholes and European put-call parity

Assume that $(S_t, t \geq 0)$ solves

$$dS_t = S_t\left(\mu dt + \sigma dW_t\right), \quad S_0 = x\,,$$

where $(W_t)_{t\in[0,T]}$ is a standard Brownian motion. Furthermore, r stands for the risk-free interest rate (with continuous compounding). We analyse some aspects of

$$\mathbb{E}\left(e^{-rT} f(S_T)\right)\,, \tag{1}$$

as being a somehow (on the first glance) a “potential candidate” for the price of a European derivative being defined by the payoff function f.

1. Write S_T as a function of W_T.
2. Compute the discounted expected payoffs of a European call (denoted by c) and of a European put (denoted by p) expressed as a function of $\mathcal{N}(d) = \int_{-\infty}^{d} e^{-\frac{x^2}{2}} \frac{dx}{\sqrt{2\pi}}$.
3. Show that the European put-call parity only holds for $\mu = r$ (for the discounted expected payoffs considered as candidates for prices).

Exercise 2: Martingale

Let $(M_t)_{t\geq 0}$ be a non-negative martingale, which satisfies $\mathbb{E}(M_t^2) < \infty$ for all $t \in [0,T]$. Show that for $s \leq t$

$$\mathbb{E}[(M_t - M_s)^2|\mathcal{F}_s] = \mathbb{E}[M_t^2 - M_s^2|\mathcal{F}_s]$$

holds.

Exercise 3: Repetition of Itô

Compute

$$d\sin W_t\,, \quad de^{W_t^2}\,, \quad de^{tW_t}\,,$$

and verify that (X_t)

$$X_t = \frac{\exp(\sigma W_t)}{1+t}$$

solves

$$dX_t = X_t \left(\frac{1}{2}\sigma^2 - \frac{1}{1+t} \right) dt + \sigma X_t dW_t$$

((W_t) is a standard Brownian motion).

Exercise 4: European power call

Assume that $(S_t, t \geq 0)$ solves

$$dS_t = S_t \left(r dt + \sigma dW_t \right), \quad S_0 = x ,$$

where (W_t) is a standard Brownian motion with respect to $\mathbb{Q}$. Compute

$$\mathbb{E}_{\mathbb{Q}}(S_T - k)_+^n .$$

Exercercice 5: Ornstein-Uhlenbeck process

Consider

$$dX_t = -cX_t dt + \sigma dW_t , \; X_0 = x. \tag{2}$$

1. Does the SDE (2) admit a unique solution?
2. Show that the solution of (2) is given by
$$X_t = e^{-ct} x + \sigma e^{-ct} \int_0^t e^{cs} dW_s.$$
3. Compute the expectation and variance of the variables X_t (Remark: The process (X_t) is a gaussian process).
4. Which is the law of X_{t+s} given $X_s = x$? Derive a simulation method $(X_{kh}, 1 \leq k \leq N)$.
5. We now assume that X_0 is a centered gaussian random variable with variance σ_0^2 being independent of (W_s). For which σ_0 we have that the law of X_t does not depend on t?