Exercises Martingales in Financial Mathematics: Model CRR and towards Black Scholes

Week 5, 2025

Exercise 1: Model CRR

The Cox Ross Rubinstein (binomial) model is sometimes used in order to price American options. Recall that there is only one risky asset with price S_n at n for all $n \in \{0, 1, \dots, N\}$ along with a risk-less asset. The risk-free interest rate for each time period is given by r. The price process (S_n) can be modeled by the relative variations of the quotes over the time subperiods, being denoted by a and b with $-1 < a < b$, i.e.

$$S_{n+1} = \begin{cases} S_n\,(1+a) \\ S_n\,(1+b) \end{cases}.$$

For our example we fix $N = 3$, $r = 0.02$, $b = 0.1$, $a = -0.1$, and $S_0 = 100$.

1. Is this particular market viable? Is it complete?
2. Describe the martingale measure $\mathbb{Q}$.
3. Draw a graph of the tree representing the possible values of S_n.
4. We intend to derive the price of an American put, where the maturity is represented by N and where the strike is given by $K = 95$. Draw a graph of the tree with the possible payoffs at n, i.e. with the used notation visualize Z_n for each possible value of S_n.
5. Starting at $n = N$, construct the Snell envelope of Z_n.

Exercise 2: The SDE of a GBM

We are interested in the following equation

$$S_t = x_0 + \mu \int_0^t S_s\,ds + \sigma \int_0^t S_s\,dW_s\,, \quad t \in [0, T]. \tag{1}$$

1. Assume

   $$S_t = x_0\, e^{\left(\mu - \frac{\sigma^2}{2}\right) t + \sigma W_t}.$$

   By using the Itô formula, show that (S_t) is a solution of (1).
2. Show that (Z_t) defined by $Z_t = x_0 / S_t$ solves

   $$Z_t = 1 + \mu' \int_0^t Z_s ds + \sigma' \int_0^t Z_s dW_s,$$

   where μ' and σ' are constants, which have to be identified.

3. Assume that (Y_t) is another solution of the SDE (1). Show that $d(Y_t Z_t) = 0$. Derive the uniqueness of the solution of (1). In finance, the unique solution of this equation is called Black and Scholes model.