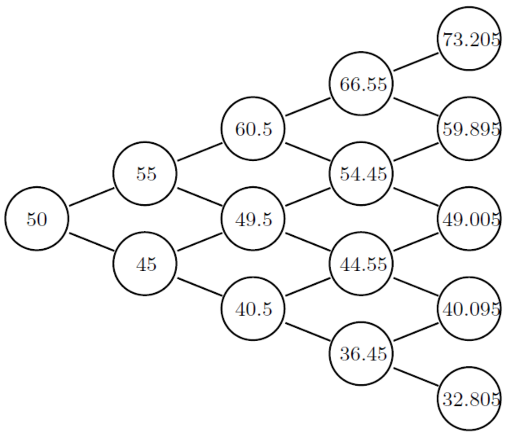
Exercises Martingales in Financial Mathematics: The complete CRR in action / on incomplete markets

Week 3, 2025

Exercise 1: Example

Consider the following binomial-tree of possible realizations for a price process, with time tick 1 month and interest rate $r_c = 0.2$ (continuous compounding), i.e. $N = 4$, $T = N\Delta t = 1/3$, $S_n^0 = e^{r_c n \Delta t}$:



(a) Calculate the value of a call with strike \$50 at time $t = 0$.

(b) Describe the replication strategy for the following scenarios:

   (i) 1. move up, 2. move down, 3. move up, 4. move up;

   (ii) 1. move down, 2. move down, 3. mote up, 4. move down.

Hint.: Use the following table:

n	move	S_n	V_n	ϕ_n	ϕ_n^0	$\tilde{V}_n$	$\tilde{S}_n$
0	-	50	?	-	-	?	50

Exercise 2: Trinomial model

The trinomial model can be considered as being an extension of the Cox Ross Rubinstein model (binomial model). There is again only one risky asset with price S_n at n until N along with a risk-less asset with risk-free interest rate r for every time period, i.e. $S_n^0 = (1+r)^n$. However, between two consecutive periods the price changes here by a factor $1+d$ or $1+m$ or $1+u$, i.e.

$$S_{n+1} = \begin{cases} S_n\,(1+d) \\ S_n\,(1+m) \\ S_n\,(1+u) \end{cases},$$

where $-1 < d < m < u$. Suppose that the initial stock price is given by S_0 and define the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with $\Omega = \{1+d, 1+m, 1+u\}^N$, $\mathcal{F} = \mathcal{P}(\Omega)$ and $\mathbb{P}$ being a probability measure such that $\mathbb{P}(\omega) > 0$ for every atom ω. Furthermore, again for $n = 1, \ldots, N$ the σ-algebra $\mathcal{F}_n$ is generated by the random variables $S_1, \ldots, S_n$, i.e. $\mathcal{F}_n = \sigma(S_1, \ldots, S_n)$ ($\mathcal{F}_0 = \{\Omega, \varnothing\}$). Finally, we define $T_n = S_n/S_{n-1}$, with possible values $1+d$, $1+m$ and $1+u$, and we assume that the T_i are i.i.d.

1. Show that in order to end up with a viable market it is necessary that $r \in]d, u[$.

2. Derive conditions for martingale measures $\mathbb{Q}$.

3. Derive that a viable market in this model is not complete.

Exercise 3: Incomplete markets

Denote by $(S_n)_{n=0,\ldots,N}$ the price vector in a viable (but not necessarily complete) market defined on a finite probability space where each element is an atom. Suppose that the random variable h defined on the same space is attainable (recall that this means that it can be replicated by an admissible strategy).

1. Show that the price V_n at n of a derivative with payoff h can be calculated uniquely by

$$V_n = S_n^0 \, \mathbb{E}^* \left[\frac{h}{S_N^0} \,|\, \mathcal{F}_n \right]$$

   where $\mathbb{E}^*$ is the expectation with respect to any measure $\widetilde{\mathbb{P}}$ under which $(\widetilde{S}_n)$ is a martingale.

2. Give an example of an incomplete market and of an attainable product in this market.