Exercises Martingales in Financial Mathematics: The CRR model

Week 2, 2025

Exercise 1: Cox Ross Rubinstein model

There is only one risky asset in the CRR model with price S_n at n until N along with a risk-less asset with risk-free interest rate r for every time period, i.e. $S_n^0 = (1+r)^n$. The risky asset is modelled as follows. Between two consecutive periods the price changes by a factor $1+a$ or $1+b$

$$S_{n+1} = \begin{cases} S_n\,(1+a) \\ S_n\,(1+b) \end{cases}$$

where $-1 < a < b$.

Suppose that the initial stock price is given by S_0 and define the probability space $(\Omega,\, \mathcal{F},\, \mathbb{P})$ with $\Omega = \{1+a,\, 1+b\}^N$, $\mathcal{F} = \mathcal{P}(\Omega)$, and $\mathbb{P}$ a probability measure such that $\mathbb{P}(\omega) > 0$ for every atom ω. For $n = 1, \ldots, N$ the σ-algebra $\mathcal{F}_n$ is generated by the random variables $S_1, \ldots, S_n$, i.e. $\mathcal{F}_n = \sigma(S_1, \ldots, S_n)$ ($\mathcal{F}_0 = \{\Omega,\, \varnothing\}$). We define the random variables $T_n = S_n/S_{n-1}$, with possible values $1+a$ and $1+b$, respectively.

1. Show that in order to end up with a viable market it is necessary that $r \in]a,\, b[$.

2. Find examples for violation of the assumption of absence of arbitrage if $r \notin]a,\, b[$.

3. Now let $r \in]a,\, b[$ and denote $p^* = (b-r)/(b-a)$. Show that $(\widetilde{S}_n)$ is a martingale under $\mathbb{Q}$ if and only if the random variables T_1, T_2, ..., T_N are i.i.d. and $\mathbb{Q}[T_1 = 1+a] = p^* = 1 - \mathbb{Q}(T_1 = 1+b)$.

4. Derive that the viable market obtained in 3 is complete (see Slides 52 and 53) and give a formula for the price of a claim with payoff H in the form of a conditional expectation with respect to $\mathbb{Q}$.

Exercise 2: Pricing of options

Continue with the notation and assumptions in the previous exercise. Furthermore, denote by C_n (P_n, respectively) the value at n of a European call (put) option with strike K and maturity N (both being written on the risky asset).

1. "Rediscover" the European put-call parity based on Point 4 of the previous exercise, i.e. derive
$$C_n - P_n = S_n - K\,(1+r)^{-(N-n)}\,.$$

2. Show that $C_n = c(n,\, S_n)$, where c is a function which can be expressed with the help of K, a, b, r and p^*.

3. Show that

$$c(n,\, x) = \frac{p^*}{1+r} c(n+1,\, x\,(1+a)) + \frac{1-p^*}{1+r} c(n+1,\, x\,(1+b)) \;\; n = 0, \ldots, N-1\,.$$

4. Show that the perfect hedging strategy of a European call at n is defined by a quantity $H_n = \Delta(n, S_{n-1})$ representing the investment in the risky asset, where the Δ is a function, which can be expressed in terms of the function c.