Martingales in Financial Mathematics

Comments on the Black-Scholes model

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Chapter 4: Some analysis of the model

— Sensitivities of the Black–Scholes formula

— Implicit volatility and model calibration

Hypotheses

The Black–Scholes model assumes

— Log-normal diffusion (restrictive, e.g. continuous).

— No bid-ask spreads.

— No taxes.

— No (other) transaction costs.

— Borrowing and lending rates are equal.

— Underlying instruments (e.g. stocks) are unlimited divisible.

— Own transactions have no influence on the price (very liquid markets).

— Complete information, which is reflected in the traded prices.

— Etc.

Recall that the Black–Scholes model

$$S_t = S_0 e^{(\mu - \frac{1}{2}\sigma^2)t + \sigma B_t}, \quad \text{i.e.} \quad dS_t = S_t(\mu dt + \sigma dB_t)$$

with a *constant* volatility σ has the weakness that really observable data tells us that σ is NOT a constant (volatility skew, volatility smile, non-constant volatility surface).

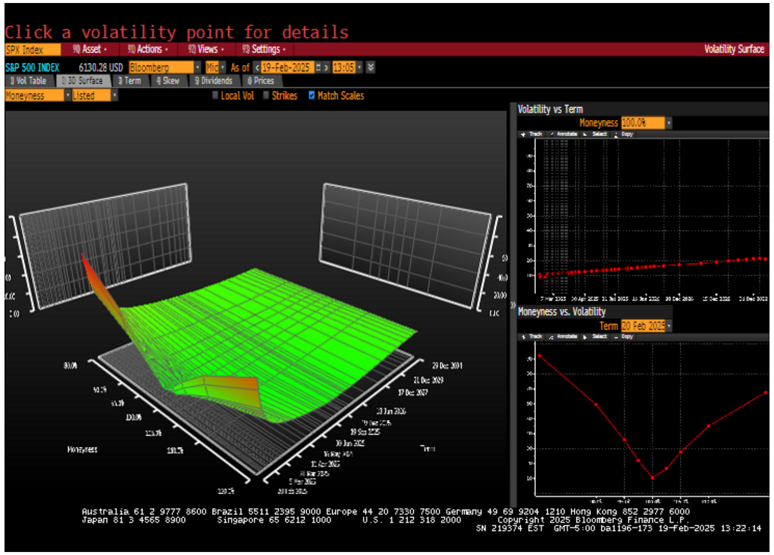

FIGURE 1 – *Source : Bloomberg*

Recall furthermore

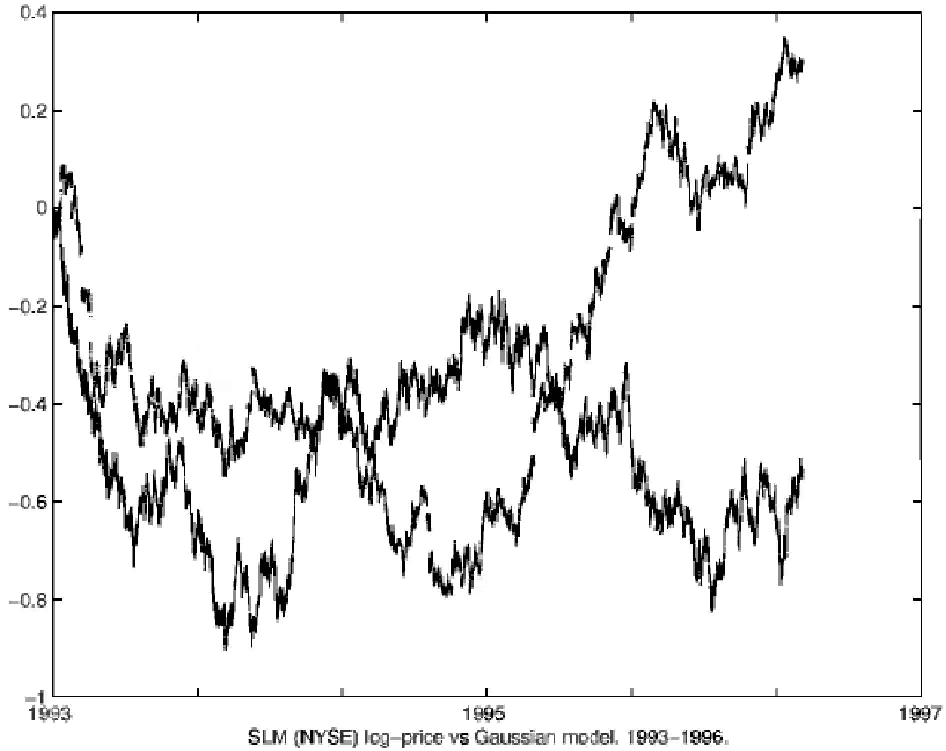

Figure 2 – *Source Cont & Tankov, Financial Modelling with Jump Processes.*

Obvious attempt to solve the problems: Model extensions

Very important possibilities: Main models based on Brownian motion

— Exponential INTEGRAL Brownian model

$$S_t = S_0 \exp\left(\int_0^t \mu_s ds + \int_0^t \sigma_s dB_s\right) .$$

— Exponential TIME-CHANGED Brownian model

$$S_t = S_0 \exp\left(\mu T(t) + B_{T(t)}\right) ,$$

i.e. $\left(B_{T(t)}\right)$ is a time change in a (standard) Brownian motion.

We will discuss that in Chapter 6.

Pragmatic attempt to tackle the problems: Via Sensitivities of an option

In particular for European call and put options the Black–Scholes setting lead to closed from formulae, depending on S_t, K, r, $T - t$, σ.

In view of the hypotheses one could (and should) be interested in the effect obtained from variations in the parameters.

IMPORTANT: *The Black–Scholes formulae have been obtained under the hypothesis that the parameters are constant, i.e. this type of analysis is heuristic (exception: the derivation of the* Δ*).*

We define the following processes (Greeks)

— Delta: Partial derivative of the pricing formula with respect to the (component of the) underlying.

— Gamma: Second partial derivative of the pricing formula with respect to the (component of the) underlying.

— Theta: $(-1)\times$ the partial derivative of the pricing formula with respect to the (component of the) time to maturity.

— Rho: Partial derivative of the pricing formula with respect to the (component of the) interest rate.

— Vega: Partial derivative with respect to the (component of the) volatility.

Some Applications

- "Greek-Matching": Besides
  - sensitivity analyses and
  - certain risk-control,

  some Greeks are sometimes used for pragmatic local ad-hoc hedges against parameter and model risks, based on other financial instruments, such as various swaps, other options, etc. (for this, of course, financial derivatives must be used, which themselves have corresponding Greeks that differ from zero).
- They can also be used for supporting
  - the model selection for concrete applications, or
  - for the selection of the specific hedging strategy (e.g. dynamic vs. semi-static, as will be explained in Exercise set 9),
  - the product design (e.g. limit certain risks in the product design).

Sensitivities of a European call

$$C_t = S_t \mathcal{N}(d_+) - K\, e^{-r(T-t)}\, \mathcal{N}(d_-)$$

$$d_\pm = \frac{\log\left(\frac{S_t}{K}\right) + \left(r \pm \frac{\sigma^2}{2}\right)\,(T-t)}{\sigma\sqrt{T-t}}$$

$$\Delta = \mathcal{N}(d_+) > 0$$

$$\Gamma = \frac{1}{\sqrt{2\,\pi}}\,\frac{1}{S_t\,\sigma\,\sqrt{T-t}}\,e^{-\frac{1}{2}d_+^2} > 0$$

$$\Theta < 0$$

$$\rho > 0$$

$$\text{Vega} = \frac{1}{\sqrt{2\,\pi}} S_t\,\sqrt{T-t}\,e^{-\frac{1}{2}d_+^2} > 0$$

Terminologie

An option is called to be

At the money if the underlying is (around) the strike.

In the money if the intrinsic value is positive.

Out of the money if the intrinsic value is negative.

The price of an option is often split in two parts

— Intrinsic value would be obtained if the option would be exercised immediately.

— Time value is the difference between the price and the intrinsic value.

Implicit volatility

Recall that the partial derivative of the pricing formula with respect to (the component of) the volatility for a European call is strictly positive. Hence, the function

$$\sigma \to C_t(S_t,\, K,\, T - t, r\ \sigma)$$

admits an inverse function.

In several markets vanilla options have been *liquidly* traded over the last decades (e.g. European options written on certain indices).

Hence, we can compute the implicit volatility “obtained from the market”, i.e. the volatility we have to plug in the Black–Scholes formula in order to arrive at the market price.

This idea can be used / seen as a simple case / idea of model calibration to traded derivatives, i.e. we derive the market parameter σ from liquidly traded derivatives in order to price new, not yet liquidly traded derivatives (risk neutral setting).

Historic versus implicit volatility

Hence, we have two approaches in order to estimate the model parameters (in the Black–Scholes setting uniquely the volatility).

— Statistic analysis, i.e. statistical approach based on historic data (time series in the real world).

— Model calibration to traded derivatives (*risk neutral* setting).

Recall that the volatility does not change if we change the measure by applying the Girsanov Theorem !

Hence, we should have $\widehat{\sigma} \approx \sigma_{IMP}$.

However, in the market, we observe $\sigma_{IMP} \gg \widehat{\sigma}$.

In view of this inconsistency we have to choose which volatility we use (if we want to price and / or hedge on the basis of a GBM). Usually, the implicit volatility is used since:

— One wants to ensure that the model leads to prices being close to the market prices.

— $\widehat{\sigma}$ is obtained from the history, where σ_{IMP} is derived from the expectations of the market participants about the future, being reflected in the market quotes.

Implicit volatility

The implicit volatility is obtained by (numerical) Black–Scholes inversion. Hence, σ_{IMP} is a function depending on other variables of the Black–Scholes formula, in particular the price of the underlying, the time to maturity, and the strike.

$$C_t \to \sigma_{IMP}(S_t, K, T - t)$$

We can fix the time(s) to maturity by fixing t. At t, S_t will be known, but often, we will have different maturity dates and, for each maturity, different strike prices (5–10 strikes for a fixed maturity is easily possible).

If the model would be consistent with the market data σ_{IMP} would be constant for all maturities and strikes. Unfortunately, this is not the case, σ_{IMP} varies depending on the time to maturity and also depending on the strike (complicate shape of the volatility surface).

Recall

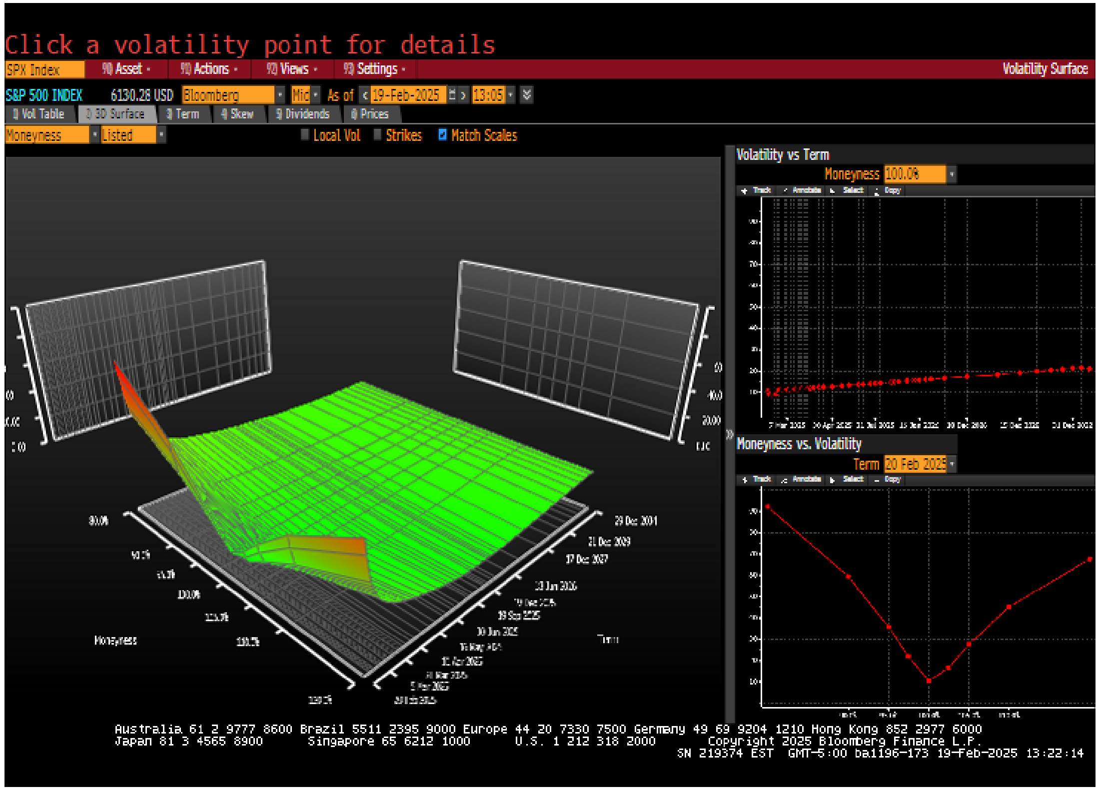

Figure 3 – *Source Bloomberg*

Outlook

There are several corrections / extensions of the Black–Scholes model:

— 1st Correction $\sigma \rightsquigarrow \sigma(t)$ (R. Merton, 1973)

— 2nd Correction $\sigma(t) \rightsquigarrow \sigma(t, S_t)$ (B. Dupire, 1994) .

As already mentioned, in Chapter 6 we will in particular discuss certain models based on:

— Stochastic volatility.

— Time changes of Brownian motion.

Hedging errors in practical implementation

Important sources for profits and losses in the implementation of hedging strategies for written products are

- Non-hedgealbe risks in the concrete theoretical model (not in Black–Scholes ; not in complete markets)
- Losses or profits from not necessarily self-financing “ad-hoc hedges” based on “Greek matching”
- Losses or profits resulting from the possible granting of certain degrees of freedom to traders
- Losses or profits from the discretization of hedging strategies and differences between market and model behaviour
- Transaction costs

In order to manage these gains and losses, Greeks often play an important role in practice (even beyond Black–Scholes).