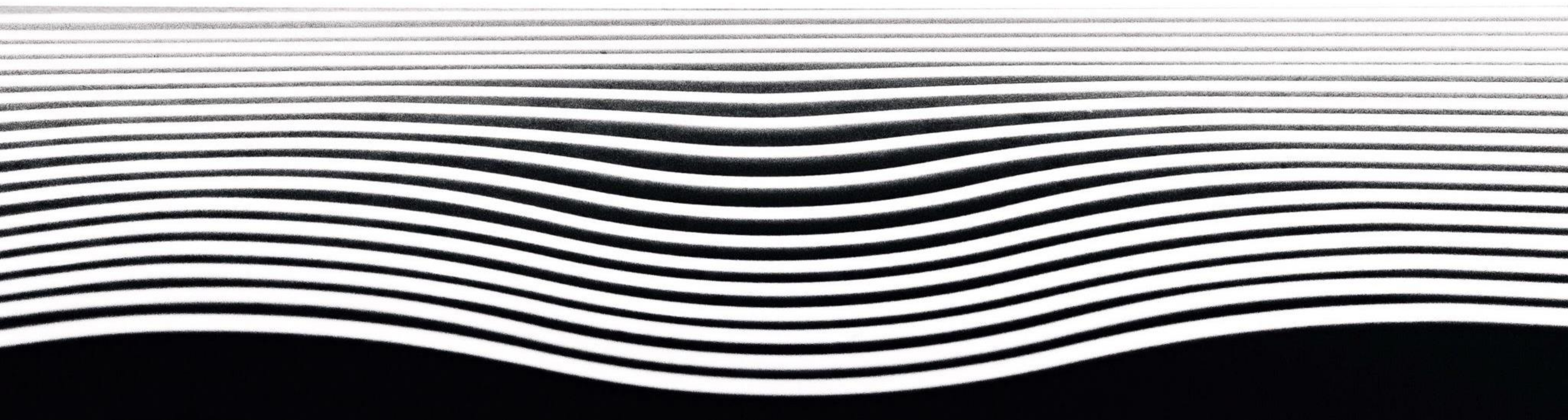
# Numerical methods for conservation laws
# 7: Conservative Finite Difference Schemes



We have seen that conservative finite difference methods look better in practice.

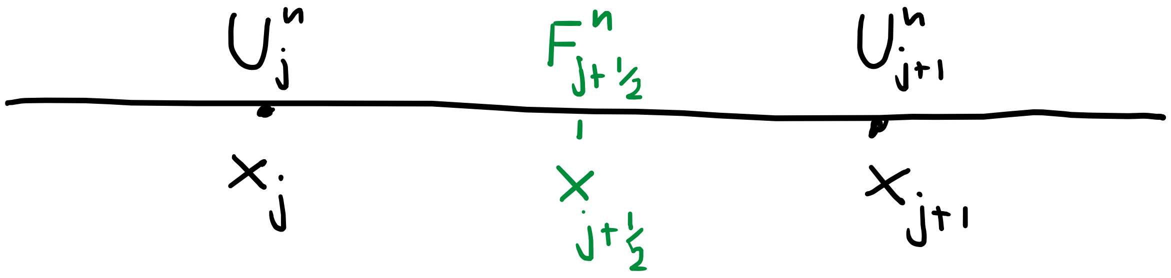
Let’s focus on such methods and their conceptual background.

We have had the idea that a finite difference scheme should be in "conservative" form.

We say a finite difference scheme is in "conservative" form if it can be written as

$$\frac{U_j^{n+1} - U_j^n}{k} + \frac{1}{h}\left( F_{j+\frac{1}{2}}^n - F_{j-\frac{1}{2}}^n \right) = 0$$

Conceptually, we define the flux at nodes in-between the nodes for $u$:



## Update formula:

$$U_j^{n+1} = U_j^n - \frac{k}{h}\left(F_{j+\frac{1}{2}}^n - F_{j-\frac{1}{2}}^n\right)$$

Schemes in conservative form satisfy the following important property

$$\sum_j U_j^{n+1} = \sum_j U_j^n - \frac{k}{h}\underbrace{\sum_j\left(F_{j+\frac{1}{2}}^n - F_{j-\frac{1}{2}}^n\right)}_{=\,?}$$

More specifically, with a telescoping sum:

$$\sum_j F^n_{j+\frac{1}{2}} - F^n_{j-\frac{1}{2}}$$

$$= F^n_{\frac{1}{2}} - F^n_{-\frac{1}{2}} + F^n_{\frac{3}{2}} - F^n_{\frac{1}{2}} + \dots + F^n_{N-\frac{1}{2}} - F^n_{N-\frac{3}{2}}$$

$$= -F^n_{-\frac{1}{2}} + F^n_{N-\frac{1}{2}} = 0 \quad \text{(assuming periodic BC)}$$

Assuming periodic boundary conditions, the system is "physically closed" and we see that the sum of the nodal values is preserved over time steps. In particular, if

$$h \sum_j U_j^n = \int_\Omega u_0(x)\, dx$$

then the integral of the initial value is preserved.

Assuming instead that the flux is constant outside of the interval of interest, we instead compute

$$\sum_j F^n_{j+1/2} - F^n_{j-1/2} = -F^n_{-1/2} + F^n_{N+1/2}$$

$$= -f(A) + f(B)$$

$$\Rightarrow \quad \sum U_j^{n+1} = \sum U_j^n - \frac{k}{h}\Big(f(B) - f(A)\Big)$$


Here, the system is generally not physically closed, there is a constant influx or outflux taking place.

This looks promising! So how do we compute the fluxes?

The numerical flux $F^n_{j+\frac{1}{2}}$ depends on the values of $U^n$ in a neighborhood of $x_{j+\frac{1}{2}}$:

$$F^n_{j+\frac{1}{2}} = F\Big( \underbrace{U^n_{j-p}, \ldots, U^n_{j-1}, U^n_j, U^n_{j+1}, \ldots, U^n_{j+q}}_{p+q+1 \text{ entries of } U^n} \Big)$$

Example

$$F^n_{j+\frac{1}{2}} = F(U^n_j) = (U^n_j)^2 \qquad \text{Burgers' Eq.}$$

$$F^n_{j+\frac{1}{2}} = F(U^n_j) = c\, U^n_j \qquad \text{Transport Eq.}$$

Some important properties of numerical fluxes that we want to be satisfied:

1. We call a numerical flux $F$ consistent if

$$F(u, u, \dots, u) = f(u)$$

This ensures that locally constant solutions stay constant.

2. Lipschitz property / first-order approximation property:

$$\left|F\left(u_1, u_2, \dots u_{p+q+1}\right) - f(u)\right| \leq M \max_{1\leq i\leq p+q+1} |u_i - u|$$

Ensures that the numerical flux approximates the physical flux.

Lipschitz implies consistent.

In practice, we only use the direct neighbors of $U_j^n$, that is, $U_{j-1}^n$ and $U_{j+1}^n$

Having introduced finite difference schemes for conservation laws, we have focused on conservative finite difference schemes.

Such schemes require that we provide a numerical flux.

Several choices of numerical fluxes are known in the literature, and now review some of them.

# Example

We evaluate the physical flux at the left or the right neighbor:

1) We use the flux $\mathrm{F}_{\mathrm{j}+\frac{1}{2}} = F(U_j) = f(U_j)$. With that:

$$U_j^{n+1} = U_j^n - \frac{k}{h}\left(\mathrm{F}_{\mathrm{j}+\frac{1}{2}} - \mathrm{F}_{\mathrm{j}-\frac{1}{2}}\right) = U_j^n - \frac{k}{h}\left(f(U_j) - f(U_{j-1})\right)$$

2) We use the flux $\mathrm{F}_{\mathrm{j}+\frac{1}{2}} = F(U_j, U_{j+1}) = f(U_j)$

$$U_j^{n+1} = U_j^n - \frac{k}{h}\left(\mathrm{F}_{\mathrm{j}+\frac{1}{2}} - \mathrm{F}_{\mathrm{j}-\frac{1}{2}}\right) = U_j^n - \frac{k}{h}\left(f(U_{j+1}) - f(U_j)\right)$$

Consistent ✓ Lipschitz ✓

# Example: Lax-Friedrichs flux

Let $\alpha = \max_u |f'(u)|$. Then we set

$$F_{LF}(u,v) = \frac{f(u)+f(v)}{2} - \frac{\alpha}{2}(v-u)$$

Typically, the parameter is only an upper estimate for $\max_u |f'(u)|$ when the we already know that $u$ stays within a certain range.

Consistent ✓ Lipschitz ✓

# Example: Lax-Wendroff flux

$$F_{LW}(u,v) = \frac{f(u)+f(v)}{2} - \frac{k}{2h} f'\left(\frac{v+u}{2}\right)(f(v)-f(u))$$

Consistent ✓ Lipschitz ✓

# Example: Roe flux

$$F_{Roe}(u,v) = \frac{f(u)+f(v)}{2} - \frac{1}{2}\left|\frac{f(v)-f(u)}{v-u}\right|(v-u)$$

Can be rewritten as:

$$F_{Roe}(u,v) = \begin{cases} f(u)\ if\ s > 0 \\ f(v)\ if\ s < 0' \end{cases} \quad \text{where} \quad s = \frac{f(u)-f(v)}{u-v}$$

Consistent ✓ Lipschitz ✓

We have introduced several fluxes for conservative finite difference schemes.

Do conservative schemes converge?

If so, to the correct solution?

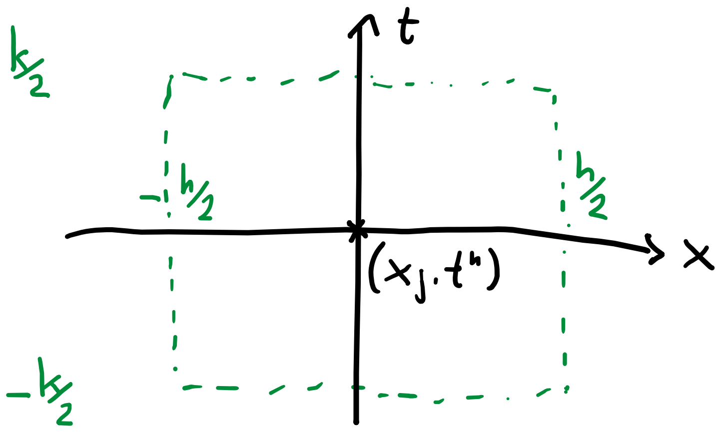
A blueprint for the latter is the Lax-Wendroff theorem, which we now discuss.

Before we discuss the convergence of finite difference schemes, we first discuss how to compare

- $u$, defined on a continuum in $(x, t)$
- $U_j^n$, defined for indices $j$ and $k$.

We define the function $U(x, t)$ as a piecewise constant function with value $U_j^n$ when

$$(x, t) \in \left[x_{j-\frac{1}{2}}, x_{j+\frac{1}{2}}\right] \times \left[t^n - \frac{k}{2}, t^n + \frac{k}{2}\right]$$



# Lax-Wendroff Theorem

Suppose we have a sequence of grids indexed by $l = 0,1,2, \ldots$ with mesh parameters $k_l, h_l \to 0$.

Let $U_l(x,t)$ be obtained with consistent and Lipschitz flux and have uniformly bounded total variation.

Suppose for every bounded set $[a,b] \times [0,T]$ we have

$$\int_0^T \int_a^b |U_l(x,t) - u(x,t)| \, \mathrm{dxdt} \to 0$$

Then $u$ is a weak solution to the conservation law.

# Proof (Lax-Wendroff Theorem)

Let $\phi \in C^1$ be a test function. We set

$$\phi_j^n = \phi(x_j, t^n)$$

The discrete scheme satisfies

$$\frac{1}{k}\left(U_j^{n+1} - U_j^n\right) + \frac{1}{h}\left(F_{j+\frac{1}{2}}^n - F_{j-\frac{1}{2}}^n\right) = 0$$

We multiply by the discrete test function and sum

$$-\sum_{j,n} \phi_j^n \frac{1}{k}\left(U_j^{n+1} - U_j^n\right) = \sum_{j,n} \phi_j^n \frac{1}{h}\left(F_{j+\frac{1}{2}}^n - F_{j-\frac{1}{2}}^n\right)$$

# Proof (Lax-Wendroff Theorem)

Since $\phi$ has compact support, the sums are finite.

We apply discrete integration by parts

$$\sum_{k=0}^{p} a^k (b^{k+1} - b^k) = -a^0 b^0 - \sum_{k=1}^{p} b^k (a^k - a^{k-1}) + a^p b^{p+1}$$

Thus

$$-\frac{1}{k} \sum_j \phi(x_j, t_0) U_j^0 - \frac{1}{k} \sum_j \sum_{n=1}^{\infty} \left( \phi(x_j, t^n) - \phi(x_j, t^{n-1}) \right) U_j^n$$

$$= \frac{1}{h} \sum_j \sum_{n=0}^{\infty} \left[ \phi(x_{j+1}, t^n) - \phi(x_j, t^n) \right] F_{j+1/2}^n$$

# Proof (Lax-Wendroff Theorem)

$$\sum_{k=0}^{p} a^k (b^{k+1} - b^k) = -a^0 b^0 - \sum_{k=1}^{p} b^k (a^k - a^{k-1}) + a^p b^{p+1}$$

$$\sum_{n=0}^{\infty} \phi_j^n (U_j^{n+1} - U_j^n) = -U_j^0 \phi_j^0 - \sum_{n=1}^{\infty} U_j^n (\phi_j^n - \phi_j^{n-1})$$

$$\sum_{j=} \phi_j^n (F_{j+1/2}^n - F_{j-1/2}^n) = -\sum_j F_{j+1/2}^n (\phi_{j+1}^n - \phi_j^n)$$

# Proof (Lax-Wendroff Theorem)

We rearrange this

$$hk \sum_j \sum_{n=0}^{\infty} \frac{\phi(x_{j+1}, t^n) - \phi(x_j, t^n)}{h} F^n_{j+1/2}$$

$$+ hk \sum_j \sum_{n=1}^{\infty} \frac{\phi(x_j, t^n) - \phi(x_j, t^{n-1})}{k} U^n_j = -h \sum_j \phi(x_j, t_0) U^0_j$$

We let $U^n_j = U_\ell(x_j, t^n)$ and observe the limit as $\ell \longrightarrow \infty$

# Proof (Lax-Wendroff Theorem)

$\boxed{1)}$

$$-h\sum_j \phi(x_j, t^0)\, U_j^0 = -\sum_j \int_{x_{j-1/2}}^{x_{j+1/2}} \phi(x_j, t^0)\, U(x, t^0)\, dx$$

Suppose that $U(x, t^0)$ is the cellwise average of $u_0$ over $[x_{j-1/2}, x_{j+1/2}]$

$$\xrightarrow[\ell \to \infty]{} \; -\sum_j \int_{x_{j-1/2}}^{x_{j+1/2}} \phi\, u_0\, dx = -\int_{-\infty}^{\infty} \phi\, u_0\, dx$$

# Proof (Lax-Wendroff Theorem)

$\boxed{2)}$

$$hk \sum_j \sum_{n=1}^{\infty} \frac{\phi(x_j, t^n) - \phi(x_j, t^{n-1})}{k} U_j^n$$

$$= \sum_j \sum_{n=1}^{\infty} \int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} \int_{t^n - \frac{1}{k}}^{t^n + \frac{1}{k}} U(x,t) \frac{\phi(x_j, t^n) - \phi(x_{j-1}, t^{n-1})}{k} \, dx\,dt$$

$$\xrightarrow[\ell \to \infty]{} \sum_j \sum_{n=1}^{\infty} \int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} \int_{t^n - \frac{1}{k}}^{t^n + \frac{1}{k}} u(x,t) \, \partial_t \phi \, dx\,dt$$

$$= \int_{-\infty}^{\infty} \int_0^{\infty} u(x,t) \, \partial_t \phi \, dx\,dt$$

# Proof (Lax-Wendroff Theorem)

$\boxed{3)}$ For the convergence of

$$hk \sum_j \sum_{n=0}^{\infty} \frac{\phi(x_{j+1}, t^n) - \phi(x_j, t^n)}{h} F^n_{j+1/2}$$

we first notice that

$$hk \sum_j \sum_{n=0}^{\infty} \frac{\phi(x_{j+1}, t^n) - \phi(x_j, t^n)}{h} f\big( U_\ell(x_j, t^n) \big)$$

$$\longrightarrow \iint \partial_x \phi \; f(u) \, dx dt$$

We thus need to estimate the difference in the flux variable!

# Proof (Lax-Wendroff Theorem)

$$F^n_{j+\frac{1}{2}} = F\left( U_\ell(x_{j-p}, t^n), \dots, U_\ell(x_{j+q}, t^n) \right)$$

$$\Rightarrow \left| f(U_\ell(x_j, t^n)) - F^n_{j+\frac{1}{2}} \right|$$

$$\leq M \cdot \max_{-p \leq i \leq q} \left| U_\ell(x_j, t^n) - U_\ell(x_{j+i}, t^n) \right|$$

Consequently, ...

# Proof (Lax-Wendroff Theorem)

... we may estimate

$$hk \sum_j \sum_{n=0}^{\infty} \frac{\phi(x_{j+1}, t^n) - \phi(x_j, t^n)}{h} F^n_{j+1/2}$$

$$= hk \sum_j \sum_{n=0}^{\infty} \frac{\phi(x_{j+1}, t^n) - \phi(x_j, t^n)}{h} f\big(U_\ell(x_j, t^n)\big) + \Delta_\ell$$

where

$$|\Delta_\ell| \le hk \sum_j \sum_{n=0}^{\infty} \max|\partial_x \phi| \cdot \max(p,q)\, M \cdot |U^n_j - U^n_{j-1}|$$

$$\le \max|\partial_x \phi| \max(p,q)\, M \cdot hk \cdot \sum_{n=0}^{\infty} TV(U^n)$$

$$\longrightarrow 0$$

$\boxed{\text{:-)}}$

Remarks:

1. The Lax-Wendroff theorem does not guarantee convergence, it assumes that convergence already holds.

2. Even if the limit exists, it is not necessarily a physical solution.

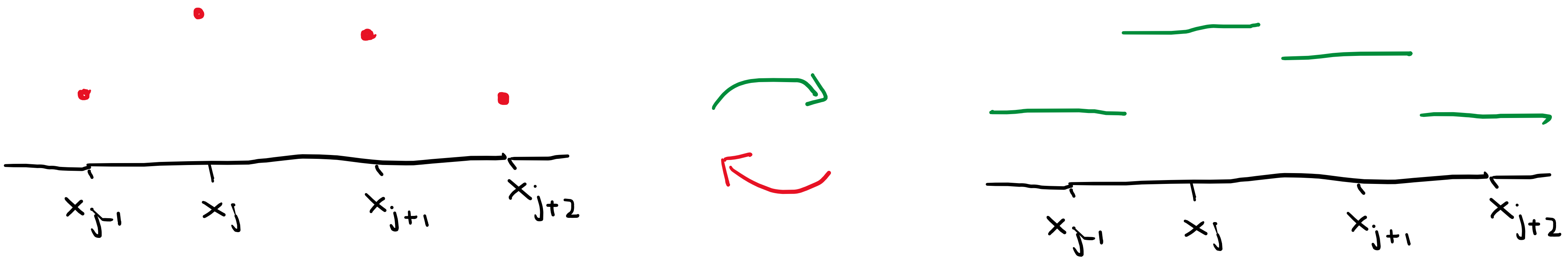
3. No statement on the order of convergence.

… we need more theory to establish anything.

# Lastly, a "dual" point of view

We have defined the discrete solution $U_j^n$ at nodal points. In particular, we are computing point values.

For comparison with any solution $u$, we have defined a piecewise constant function $U(x,t)$.




Instead of thinking in terms of values at nodes $x_j$, we may think of averages within "buckets" $\left[x_{j-\frac{1}{2}}, x_{j+\frac{1}{2}}\right]$.

This point of view leads to finite volume methods.

More about that … soon!