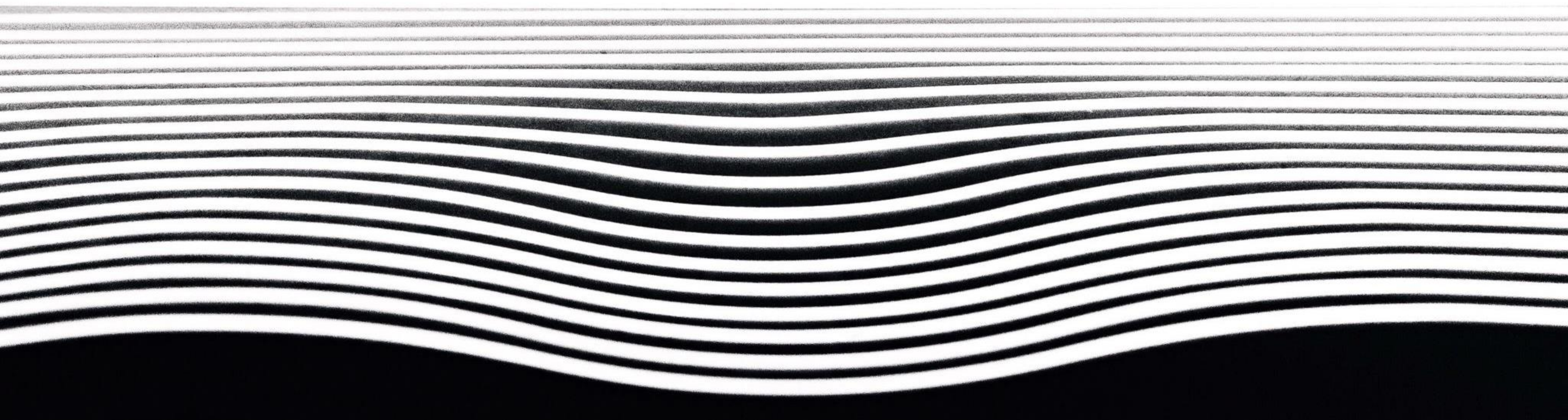
Numerical methods for conservation laws
13: Systems in Higher Dimensions



We have discussed systems of conservations in 1D and their numerical analysis.

Let us briefly address systems in higher dimensional space and their numerical analysis.

We study systems of the form

$$\partial_t U + \operatorname{div} f(U) = 0$$

where

$$U = \begin{pmatrix} U_1 \\ U_2 \\ \vdots \\ U_m \end{pmatrix} \qquad f(U) = \begin{pmatrix} | & | & & | \\ f_1 & f_2 & \cdots & f_d \\ | & | & & | \end{pmatrix} \in \mathbb{R}^{m \times d}$$

Each component U_1 depends on t and $x \in \mathbb{R}^d$

Example: Linear Systems [Evans Ch 7.3]

$$\partial_t U + \sum_{i=1}^{d} A_i \, \partial_{x_i} U = 0$$

Standard form of a linear first-order hyperbolic system

What is the flux ?

$$F(U) = \left(A_1 U, \; A_2 U, \; \ldots, \; A_d U \right)$$

$$\operatorname{div} F(U) = A_1 \cdot \partial_{x_1} U + A_2 \cdot \partial_{x_2} U + \cdots + A_d \cdot \partial_{x_d} U$$

Example Fluid Dynamics in N-D, $m = d$

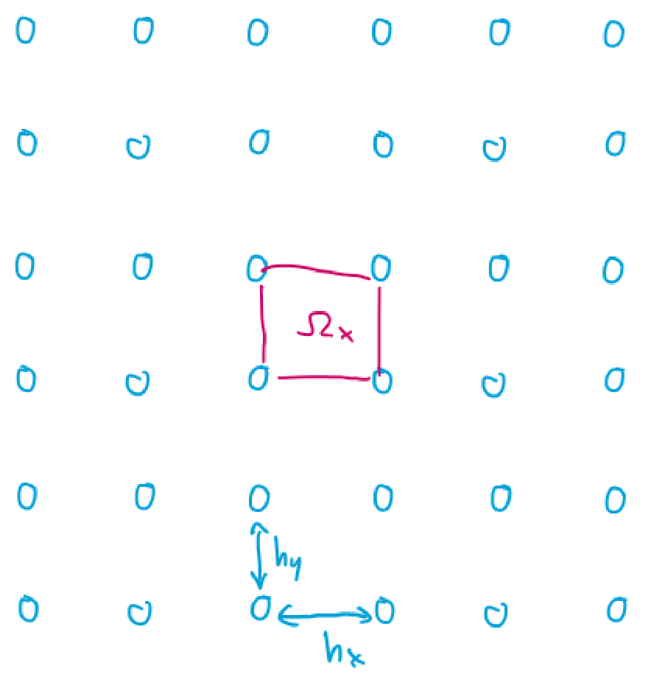
$$\partial_t U + \frac{1}{2} \operatorname{div}\left(U \cdot U^T \right) = 0$$

$\mathbb{R}^{d\times d}$

$$U \cdot U^T = \begin{pmatrix} U_1 U_1 & U_1 U_2 & \cdots & U_1 U_m \\ U_2 U_1 & U_2 U_2 & & U_2 U_m \\ \vdots & \vdots & & \vdots \\ U_m U_1 & U_m U_2 & & U_m U_m \end{pmatrix}$$

$U \cdot U^T$ is also written $U \otimes U$ in some sources

Suppose we work over a Cartesian grid



$$\Omega_x = \underbrace{[x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}}]}_{h_x} \times \underbrace{[y_{j-\frac{1}{2}}, y_{j+\frac{1}{2}}]}_{h_y}$$

We study the conservation law over the cell Ω_x, and integrate over the time interval Ω_t.

$$\Omega_t = [t_1, t_2]$$

$$0 = \int_{\Omega_t} \int_{\Omega_x} \partial_t U + \operatorname{div} f(U) \, dx\,dt$$

Applying the Divergence theorem:

$$0 = \int_{\Omega_x} U(x,t_2) - U(x,t_1) \, dx + \int_{\Omega_t} \oint_{\partial\Omega_x} \vec{n} \cdot f(U) \, dx\,dt$$

outer unit normal along $\partial\Omega_x$

$$\oint_{\partial\Omega_x} \vec{n} \cdot F(U)\, ds = \int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} f(U(x, y_{j+\frac{1}{2}}, t))_y - f(U(x, y_{j-\frac{1}{2}}, t))_y \, dx$$

$$+ \int_{y_{j-\frac{1}{2}}}^{y_{j+\frac{1}{2}}} f(U(x_{i+\frac{1}{2}}, y, t))_x - f(U(x_{i-\frac{1}{2}}, y, t))_x \, dy$$

We define a numerical method via approximating the flux integrals and computing averages.

We define cell averages

$$\bar{U}^n_{ij} = \frac{1}{h_x h_y} \int_{\Omega_{ij}} U(x,y,t)\, dx\, dy$$

and approximate the flux using a midpoint rule in x

$$\oint_{\partial\Omega_{ij}} \vec{n} f(U) \approx h_x \Big(f(U^*(x_i, y_{j+\frac{1}{2}}, t))_y - f(U^*(x_i, y_{j-\frac{1}{2}}, t))_y \Big)$$

$$+\, h_y \Big(f(U^*(x_{i+\frac{1}{2}}, y_j, t))_x - f(U^*(x_{i-\frac{1}{2}}, y_j, t))_x \Big)$$

$$\int_{\Omega_x} U(x,t_2) = \int_{\Omega_x} U(x,t_1)\,dx - \int_{t_1}^{t_2} \oint_{\partial\Omega_x} \vec{n}\cdot f(U)\,dx$$

Substitute, use left point rule in t, divide by $h_x h_y$:

$$\bar{U}_{ij}^{n+1} = \bar{U}_{ij}^{n} - \frac{k}{h_y}\Big(f(U^*(x_i, y_{j+\frac12}, t_n))_y - f(U^*(x_i, y_{j-\frac12}, t_n))_y \Big)$$

$$- \frac{k}{h_x}\Big(f(U^*(x_{i+\frac12}, y_j, t_n))_x - f(U^*(x_{i-\frac12}, y_j, t_n))_x \Big)$$

A similar construction is possible in 3D dimensions, with the obvious modifications. More generally, we generalize this setup to unstructured cellular meshes, with further additional complexity.

This finishes our outline of systems of conservations.

Now back to scalar conservation laws.