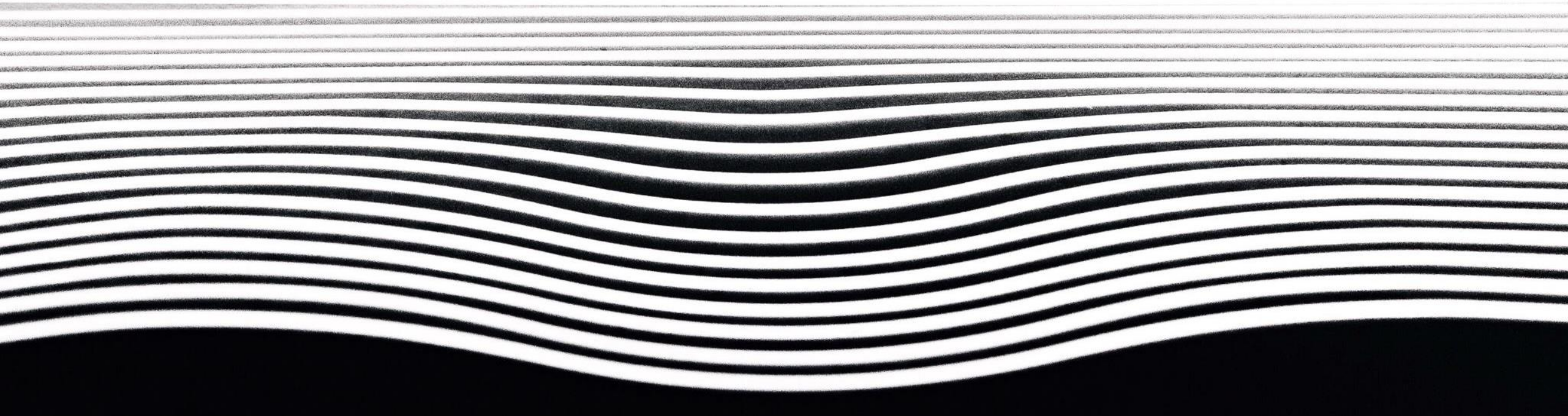
Numerical methods for conservation laws
12: Finite volume methods for systems of conservation laws



Recap

Previously, we have discussed finite volume methods for scalar conservation laws.

We have also discussed systems of conservation laws in 1D.

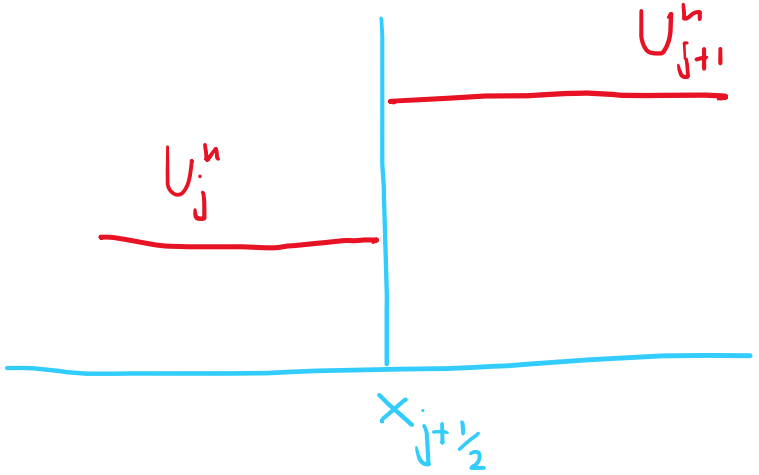
In this lecture, we apply finite volume methods to systems in 1D.

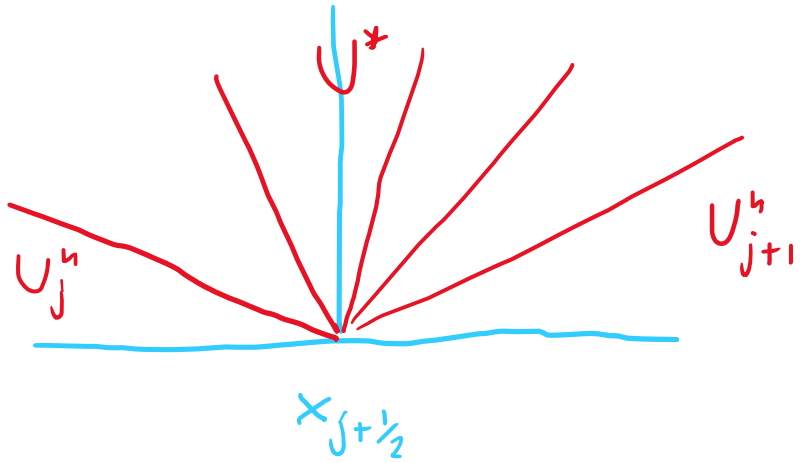
The basic setup is the same as in the scalar case, so we skip to the interesting part and address Gudonov's method.

We first address linear systems.

We revisit the flux integral in the FVM.

$$\partial_t U + \underline{\underline{A}} \cdot \partial_x U = 0$$





We want to compute $\int_{t^n}^{t^{n+1}} f(x_{j+1/2}, s)\, dt$ (approximately)

Suppose we have the eigenvalues and two initial states

$$\lambda_1 > \cdots > \lambda_q > 0 > \lambda_{q+1} > \cdots > \lambda_m$$

$$\underline{\bar{U}}_j = \underline{\underline{S}}\,\underline{\alpha} \quad , \qquad \underline{\bar{U}}_{j+1} = \underline{\underline{S}}\,\underline{\beta}$$

We find

$$\underline{U}(x_{j+\frac{1}{2}}, t) = \sum_{i=1}^{q} \alpha_i \underline{S}_i + \sum_{i=q+1}^{m} \beta_i \underline{S}_i$$

$$= \underline{\bar{U}}_j + \sum_{i=q+1}^{m} \underbrace{(\beta_i - \alpha_i)}_{\gamma_i} \underline{S}_i$$

$$= \underline{\bar{U}}_{j+1} - \sum_{i=1}^{q} (\beta_i - \alpha_i) \underline{S}_i$$

Now we find that

$$\underline{U}(x_{j+\frac{1}{2}}, t) = \bar{\underline{U}}_j + \sum_{i\,:\,\lambda_i<0} \alpha_i \underline{S}_i = \bar{\underline{U}}_{j+1} - \sum_{i\,:\,\lambda_i>0} \alpha_i \underline{S}_i$$

Thus the flux becomes

$$\underline{\underline{A}}\,\underline{U}(x_{j+\frac{1}{2}}, t) = \underline{\underline{A}}\bar{\underline{U}}_j + \sum_{i\,:\,\lambda_i<0} \alpha_i \lambda_i \underline{S}_i$$

$$= \underline{\underline{A}}\bar{\underline{U}}_{j+1} - \sum_{i\,:\,\lambda_i>0} \alpha_i \lambda_i \underline{S}_i$$

Note that

$$\underline{\underline{A}}\left(\bar{\underline{U}}_{j+1} - \bar{\underline{U}}_j\right) = \sum_{i=1}^{m} \alpha_i \lambda_i \underline{S}_i$$

$$\underline{\underline{A}} \left(\underline{U}(x_{j+\frac{1}{2}}, t) - \bar{\underline{U}}_j \right) = \sum_{i : \lambda_i < 0} \alpha_i \lambda_i \underline{S}_i$$

$$\underline{\underline{A}} \left(\underline{U}(x_{j+\frac{1}{2}}, t) - \bar{\underline{U}}_{j+1} \right) = -\sum_{i : \lambda_i > 0} \alpha_i \lambda_i \underline{S}_i$$

We split the diagonal matrix Λ into a positive and a negative part: $\Lambda = \Lambda^+ + \Lambda^-$

$$\Lambda^+_{ii} = \max(0, \lambda_i), \qquad \Lambda^-_{ii} = \min(0, \lambda_i)$$

$$\Lambda^+ = \begin{pmatrix} \lambda_1 & & & \\ & \ddots & & \\ & & \lambda_q & \\ & & & 0 \\ & & & & \ddots \\ & & & & & 0 \end{pmatrix}, \qquad \Lambda^- = \begin{pmatrix} 0 & & & \\ & \ddots & & \\ & & 0 & \\ & & & \lambda_{q+1} \\ & & & & \ddots \\ & & & & & \lambda_m \end{pmatrix}$$

We get

$$A^+ = S\Lambda^+ S^{-1}, \qquad A^- = S\Lambda^- S^{-1}, \qquad |A| = A^+ - A^-$$

$$|A| = S|\Lambda|S^{-1}$$

With that,

$$\underline{\underline{A}} \cdot \underline{U}(x_{j+\frac{1}{2}}, t) = \underline{\underline{A}}\,\underline{\bar{U}}_j + A^-\left(\underline{\bar{U}}_{j+1} - \underline{\bar{U}}_j\right)$$

$$= \underline{\underline{A}}\,\underline{\bar{U}}_{j+1} - A^+\left(\underline{\bar{U}}_{j+1} - \underline{\bar{U}}_j\right)$$

Adding these gives

$$\underline{\underline{A}} \cdot \underline{U}(x_{j+\frac{1}{2}}, t) = \frac{1}{2}\underline{\underline{A}}\left(U_{j+1} + U_j\right) + \frac{1}{2}\left(A^- - A^+\right)\left(\underline{\bar{U}}_{j+1} - \underline{\bar{U}}_j\right)$$

$$= \frac{1}{2}\underline{\underline{A}}\left(U_{j+1} + U_j\right) - \frac{1}{2}|A|\left(\underline{\bar{U}}_{j+1} - \underline{\bar{U}}_j\right)$$

Thus

$$\underline{\underline{A}} \cdot \underline{U}(x_{j+\frac{1}{2}}, t) = \frac{1}{2}\underline{\underline{A}}\left(U_{j+1} + U_j\right) - \frac{1}{2}|A|\left(\bar{U}_{j+1} - \bar{U}_j\right)$$

Notice $f(U) = \underline{\underline{A}} \cdot U$ and $\|\underline{\underline{A}}\|_2 \leq \lambda_{max} := \max(\lambda_1, -\lambda_m)$

This inspires the Lax-Friedrichs flux once more

$$F_{j+\frac{1}{2}} := \frac{f(\bar{U}_j) + f(\bar{U}_{j+1})}{2} - \frac{|\lambda_{max}|}{2}\left(\bar{U}_{j+1} - \bar{U}_j\right)$$

So much for Gudonov's method applied to linear systems but what about nonlinear systems?

Two major concerns:

1. At each time step, we can try to solve the Riemann problems exactly and then average the solution. But that last step already introduces some error, hence solving the local problems exactly might not be necessary in the first place.

2. Even if we want to use exact solution to the local Riemann problems in Gudonov's method, that might be generally impractical, in particular with a nonlinear flux.

There are two basic approaches to adapt Gudonov's method:

a) Approximate the intermediate state $U^* \approx U^*(U_L, U_R)$ and compute the flux integrals

$$\int_{t^n}^{t^{n+1}} f(\, U^*(U_L, U_R\,)\; ds$$

The resulting method, by construction, be conservative.

b) Alternatively, we solve the local Riemann problems only approximately, and then average the solution.

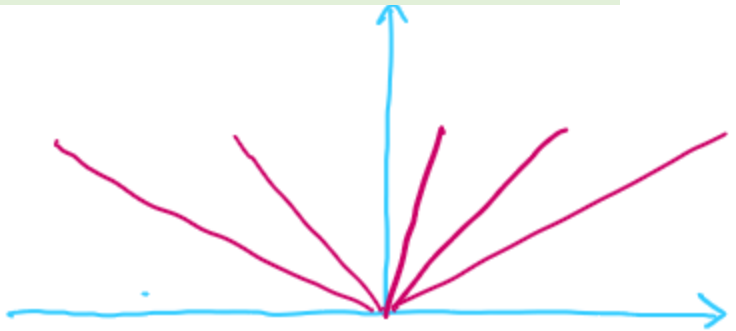
We focus on the second option: solve the local Riemann problems only inexactly.

In most cases, we replace the original conservation law by a simplified conservation law that we do solve exactly.

We linearize the flux with yet to be defined linearization $A^*(U_L, U_R)$

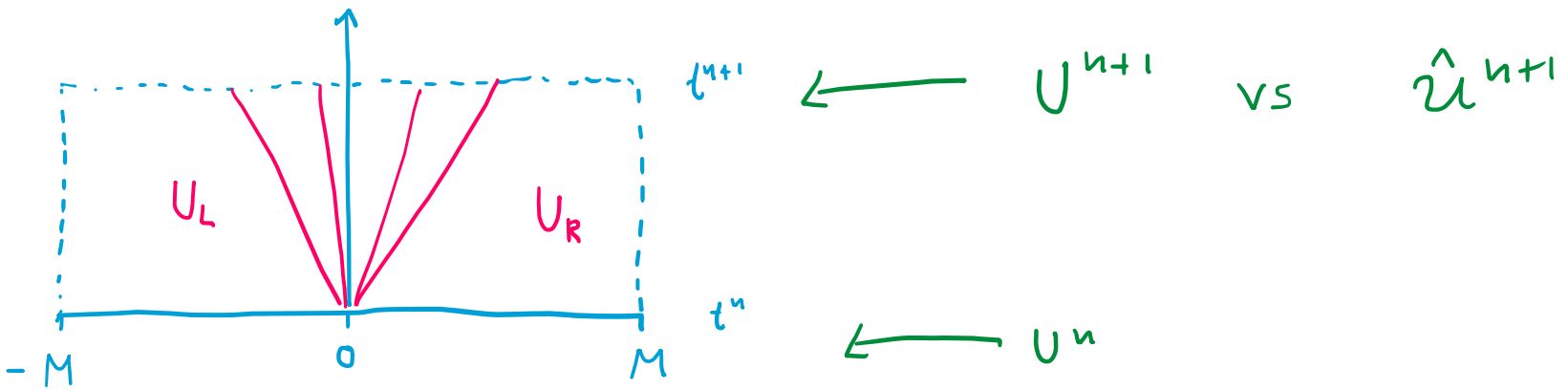
$$A(U) \approx A^*(U_L, U_R),$$

We have m discontinuities and $m - 1$ intermediate states.

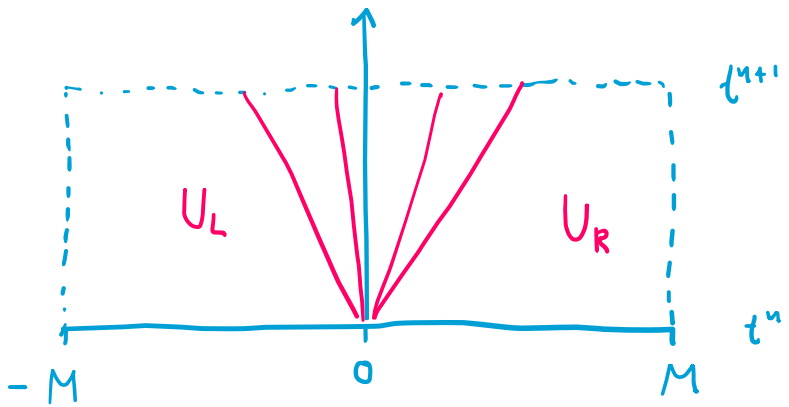


So suppose we solve a local Riemann problem with a linearized flux A^*.

Given local initial data u^n at time t^{n+1}
let u^{n+1} solve the original Riemann problem at time t^{n+1},
and let $\hat{u}^{n+1}$ solve the approximate problem at time t^{n+1}



Suppose we solve a local Riemann problem between times t^n and t^{n+1}



Conservation:

$$\int_{-M}^{M} \hat{u}(x, t^{n+1})\,dx = M(U_L + U_R) + \int_{t^n}^{t^{n+1}} f(U_L) - f(U_R)$$

That should remain true for any approximate Riemann solver!

What does that mean for the approximate flux?

Since

$$\int_{-M}^{M} u^{n+1}(x, t^{n+1}) = \int_{-M}^{M} u^{n}(x, t^{n}) + \int_{t^n}^{t^{n+1}} f(U_L) - f(U_R)$$

§§

$$\int_{-M}^{M} \hat{u}^{n+1}(x, t^{n+1}) = \int_{-M}^{M} u^{n}(x, t^{n}) + \int_{t^n}^{t^{n+1}} A^*(U_L - U_R)$$

the approximate linear flux should satisfy

$$A^*(U_L - U_R) \approx f(U_L) - f(U_R)$$

The original flux f_0 at the origin satisfies

$$\int_0^M u(x, t^{n+1})\,dx = \int_{t^n}^{t^{n+1}} f_0 - f(U_R)\,ds + M \cdot U_R$$

The approximate flux $f(U_L, U_R)$ must satisfy

$$\int_0^M \hat{u}(x, t^{n+1})\,dx = \int_{t^n}^{t^{n+1}} F(U_L, U_R) - f(U_R)\,ds + M \cdot U_R$$

$$\textcircled{*}_R \quad f_0 \approx f(U_L, U_R) = f(U_R) - \sum_{\lambda_i > 0} \lambda_i^* \gamma_i^* \underline{S}_i$$

The original flux f_0 at the origin satisfies

$$\int_{-M}^{0} u(x, t^{n+1}) \, dx = \int_{t^n}^{t^{n+1}} f(U_L) - f_0 \, ds + M \cdot U_L$$

The approximate flux $f(U_L, U_R)$ must satisfy

$$\int_{-M}^{0} \hat{u}(x, t^{n+1}) \, dx = \int_{t^n}^{t^{n+1}} f(U_L) - f(U_L, U_R) \, ds + M \cdot U_L$$

$$\circledast_L \quad f_0 \approx f(U_L, U_R) = f(U_L) + \sum_{\lambda_i < 0} \lambda_i^* \gamma_i \underline{S}_i$$

What conditions do we need for A^* when solving the local linearized problem?

$$\partial_t u + A^*(U_L, U_R)\, \partial_x U = 0$$

1) A^* must be diagonalizable with real eigenvalues

preserves hyperbolicity

2) $A^*(U_L, U_R) \longrightarrow \nabla_U f(U)$ as $U_L, U_R \longrightarrow U$

3) $A^*(U_L, U_R)\left(U_R - U_L\right) = f(U_R) - f(U_L)$

(Roe condition)

An observation about the Roe condition:

If the solution to the original Riemann problem is a shock wave with speed s, then the Rankine-Hugoniot condition implies

$$f(U_L) - f(U_R) = s(U_L - U_R)$$

We want that simple special case to be replicated exactly. Thus

$$A^*(U_L - U_R) = s(U_L - U_R)$$

In other words, $U_L - U_R$ must be an eigenvector of the linearized flux and the shock speed must be an eigenvalue of A^* .

Q: Does a flux with such properties exist?

Given the linearized flux, we pretend the local system is linear. In the FV scheme we use

$$F_{j+1/2} = \frac{1}{2}\left(f(U_L) + f(U_R) \right) - \frac{1}{2}|A^*|\left(U_L - U_R \right)$$

It remains to find the matrix $|A^*|$

We do we compute the linearized flux?

a) Suppose we have $U : [0,1] \longrightarrow \mathbb{R}$, $U(0) = U_L$, $U(1) = U_R$

Consider $$f(U_R) - f(U_L) = \int_0^1 \partial_\xi f(U(\xi))\, d\xi$$

$$= \int_0^1 \nabla f(U(\xi))\, \partial_\xi U\, d\xi$$

For simplicity, assume "Roe linearization"

$$U(\xi) = U_L + \xi(U_R - U_L), \qquad \partial_\xi U = U_R - U_L$$

Then

$$f(U_R) - f(U_L) = \underbrace{\int_0^1 \nabla f(U(\xi))\, d\xi}_{\text{"Optimal" choice for } A^*(U_L, U_R)} \cdot (U_R - U_L)$$

Hard to compute

We can try to use

$$A^*(U_L, U_R) := \int_0^1 \nabla f(U(\xi))\, d\xi$$

Generally not computable

b) Numerical quadrature

$$A^*(U_L, U_R) := \nabla f(U(1/2)) = \nabla f\left(\frac{U_L + U_R}{2}\right)$$

Roe condition not guaranteed :-(

c) Numerical quadrature

$$A^*(U_L, U_R) := \frac{1}{2}\left(\nabla f(U(0)) + \nabla f(U(1))\right)$$

$$= \frac{1}{2}\left(\nabla f(U_L) + \nabla f(U_R)\right)$$

Hyperbolicity not guaranteed

We reconsider the Roe condition from a different angle. The Roe condition reflects the presence of one dominant discontinuity with speed s.

Recall the Rankine-Hugoniot condition

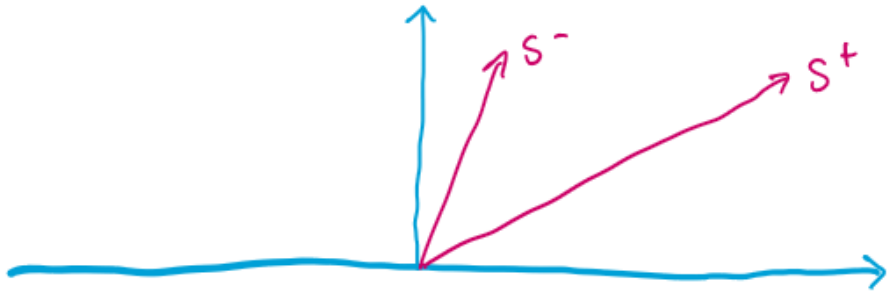
$$f(U_L) - f(U_R) = s(\, U_L - U_R)$$

The Roe condition leads to

$$A^*(U_L - U_R) = s\,(\, U_L - U_R)$$

In other words, the Roe condition reflects the assumption of one strong discontinuity

We improve this by incorporating **two** strong shocks, the one with fast speed s^+ and the one with slowest speed s^-



HLL flux (Harten, Lax, van Leer)

Assuming knowledge on the speed s^- and s^+, we set

$$A^*(U_L, U_R) = \begin{cases} f(U_L) & \text{if } s^- > 0 \quad (++) \\ \dfrac{s^+ f(U_L) - s^- f(U_R)}{s^+ - s^-} + \dfrac{s^+ s^-}{s^+ - s^-}(U_L - U_R) & \text{if } s^- < 0 < s^+ \quad (-+) \\ f(U_R) & \text{if } s^+ < 0 \quad (--) \end{cases}$$