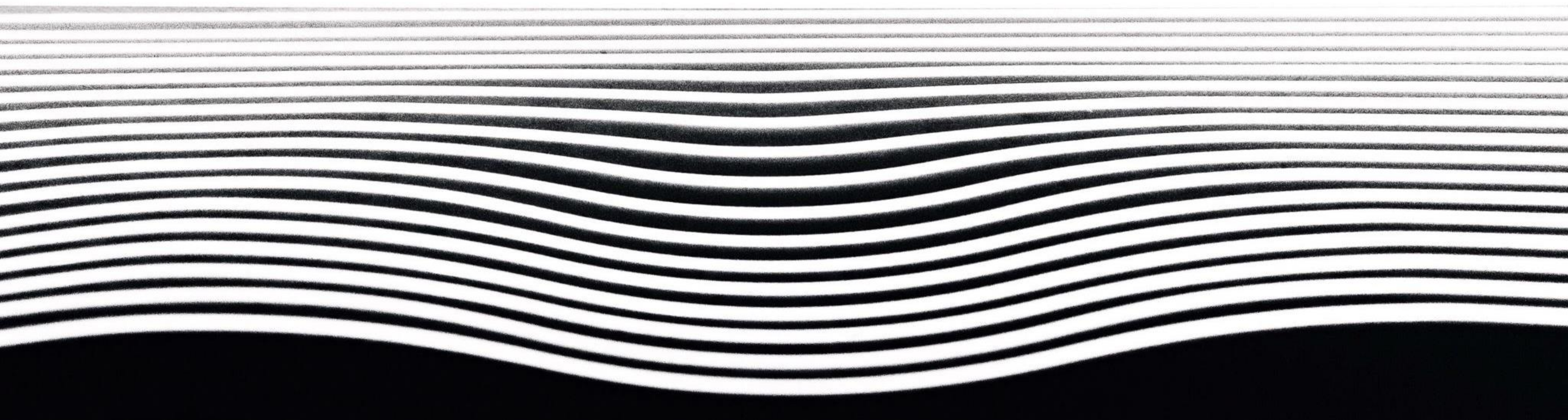
Numerical methods for conservation laws
11: Systems of Conservation Laws



Recap

We have had our focus on scalar conservation laws up to this point.

We now address the solution theory of systems of conservation laws.

We begin the discussion with a linear system

$$\partial_t \underline{U} + \underline{\underline{A}} \cdot \partial_x \underline{U} = 0$$

matrix of size $m \times m$

$$\underline{U} = (U_1, U_2, \ldots, U_m)^T$$

m functions in x and t

We call the system hyperbolic if the eigenvalues of $\underline{\underline{A}}$ are real

We call the system strongly hyperbolic if $\underline{\underline{A}}$ is diagonalizable, otherwise weakly hyperbolic

Moreover, we call it strictly hyperbolic if the eigenvalues of $\underline{\underline{A}}$ are distinct

If A is symmetric, we call it symmetric hyperbolic.

Important vocabulary in different languages:

Eigenvalue / Eigenwert / egenverdi / eigenwaarde / 特征值 / valeur propre

Eigenvector / Eigenvektor / Egenvektor / eigenvector / 特徵向量 / vecteur propre

For simplicity we assume that A is diagonalizable.

$$\underline{\underline{A}} = \underline{\underline{S}}\ \underline{\underline{\Lambda}}\ \underline{\underline{S}}^{-1} \qquad \underline{\underline{S}} \in \mathbb{R}^{m \times m} \text{ invertible}, \quad \underline{\underline{\Lambda}} \text{ diagonal}$$

The diagonal entries of $\underline{\underline{\Lambda}}$ are the eigenvalues of $\underline{A}$ and the columns $\underline{S}_i$ are the eigenvectors.

$$\underline{\underline{\Lambda}} = \begin{pmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_m \end{pmatrix} \qquad \underline{\underline{S}} = \begin{pmatrix} | & | & & | \\ s_1 & s_2 & \cdots & s_m \\ | & | & & | \end{pmatrix}$$

$$\lambda_1 > \lambda_2 > \cdots > \lambda_m$$

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Assuming that $\underline{\underline{A}}$ is constant, we rewrite the conservation law

$$\partial_t \underline{U} + \underline{\underline{A}}\ \partial_x \underline{U} = 0$$

$$\Rightarrow \quad \underline{\underline{S}}^{-1} \partial_t \underline{U} + \underline{\underline{S}}^{-1} \underline{\underline{A}}\ \underline{\underline{S}}\ \underline{\underline{S}}^{-1} \partial_x \underline{U} = 0$$

$$\Rightarrow \quad \partial_t \underline{V} + \underline{\underline{\Lambda}}\ \partial_x \underline{V} = 0$$

Example

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$$\Rightarrow \quad \partial_t \underline{V} + \underline{\underline{\Lambda}}\, \partial_x \underline{V} = 0$$

1. Multiply by $\underline{\underline{S}}^{-1}$
2. $\mathrm{Id} = \underline{\underline{S}}\, \underline{\underline{S}}^{-1}$
3. $V = \underline{\underline{S}}^{-1} \underline{U}$

Those are m independent linear transport equations. The initial condition of the system is

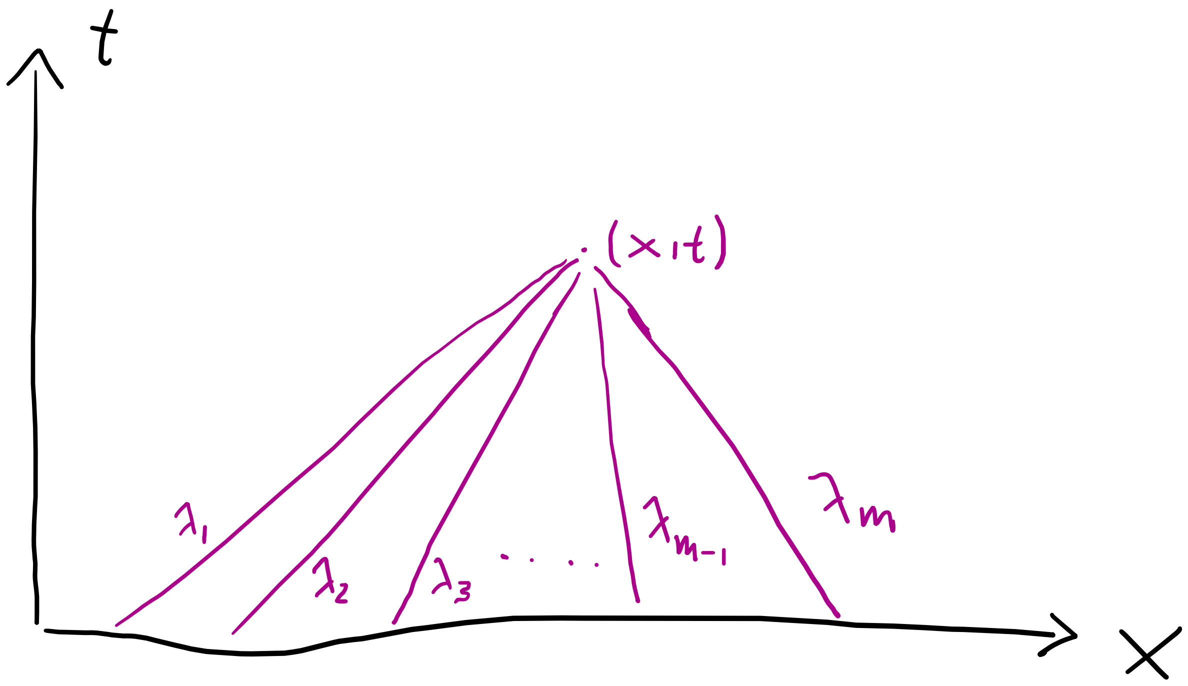
$$\underline{V}_0(x) = \underline{\underline{S}}^{-1} \cdot \underline{U}_0(x)$$

i-th eigenvalue of $\underline{\underline{A}}$

In each component: $\quad v_i(x,t) = v_{i,0}(x - \lambda_i t)$

Together: $\quad \underline{U}(x,t) = \underline{\underline{S}} \cdot \underline{V}(x,t) = \sum_{i=1}^{m} v_i(x,t)\, \underline{S}_i$

The exact solution is a sum of m waves that propagate with respective speeds λ_i , and we can solve each of them separately.



Domain of Dependence

$\lambda_1 > \lambda_2 > \dots > \lambda_{m-1} > \lambda_m$

Example: Linear Wave equation

$$U_{tt} - U_{xx} = 0 \iff U_t + V_x = 0, \quad V_t + U_x = 0$$

$$\begin{bmatrix} U \\ V \end{bmatrix}_t + \underbrace{\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}}_{= \underline{\underline{A}}} \begin{bmatrix} U \\ V \end{bmatrix}_x = 0$$

We have

$$\underline{\underline{A}} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \underline{\underline{S}} \cdot \underline{\underline{\Lambda}} \cdot \underline{\underline{S}}^{-1}$$

with

$$\underline{\underline{\Lambda}} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad \underline{\underline{S}} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = \underline{\underline{S}}^{-1}$$

The solution is a superposition of two waves propagating into opposite directions.

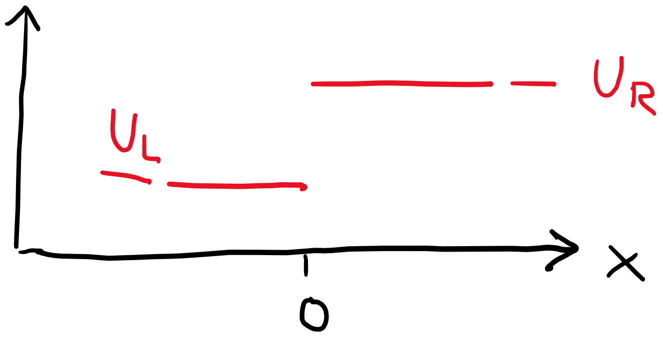
$$\begin{bmatrix} U \\ V \end{bmatrix} = w_1(x,t) \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix} + w_2(x,t) \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix}$$

$$w_1(x,t) = w_{1,0}(x-t)$$

$$w_2(x,t) = w_{2,0}(x+t).$$

We consider a Riemann problem for the system of conservations laws, with two initial states U_L and U_R.

$$U_0(x) = \begin{cases} U_L & x < 0 \\ U_R & x > 0 \end{cases}$$



$$U_L = S\alpha = \sum_{i=1}^{m} \alpha_i S_i \qquad \alpha = (\alpha_1, \alpha_2, \dots, \alpha_m)$$

$$U_R = S\beta = \sum_{i=1}^{m} \beta_i S_i \qquad \beta = (\beta_1, \beta_2, \dots, \beta_m)$$

We switch to the “characteristic coordinate system” of the state variable, in which the solution V is described by the coordinates v_i

$$V_0 = \begin{pmatrix} V_{1,0} \\ V_{2,0} \\ \vdots \\ V_{m,0} \end{pmatrix} \qquad V_{i,0} = \begin{cases} \alpha_i & x < 0 \\ \beta_i & x > 0 \end{cases}$$

$$V(x,t) = \begin{pmatrix} V_1 \\ V_2 \\ \vdots \\ V_m \end{pmatrix} \qquad V_i(x,t) = \begin{cases} \alpha_i & x < \lambda_i t \\ \beta_i & x > \lambda_i t \end{cases}$$

Example:

$$\underline{\underline{A}} = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{pmatrix}$$

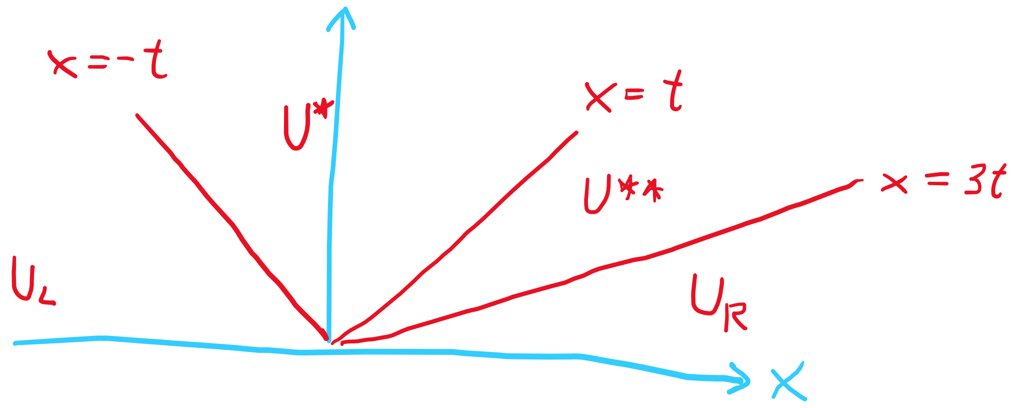
$\lambda_1 = 3, \quad \lambda_2 = 1, \quad \lambda_3 = -1$

$$S_1 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \quad S_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \quad S_3 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$$

Consider the case

$$U_L = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}, \quad U_R = \begin{pmatrix} 0 \\ 0 \\ -2 \end{pmatrix} \quad \Rightarrow \quad \alpha = \begin{pmatrix} \sqrt{2} \\ 1 \\ \sqrt{2} \end{pmatrix}, \quad \beta = \begin{pmatrix} -\sqrt{2} \\ 0 \\ \sqrt{2} \end{pmatrix}$$

The solution looks as follows:



$$\begin{aligned} U_L &= \alpha_1 S_1 + \alpha_2 S_2 + \alpha_3 S_3 \\ U^* &= \alpha_1 S_1 + \alpha_2 S_2 + \beta_3 S_3 \\ U^{**} &= \alpha_1 S_1 + \beta_2 S_2 + \beta_3 S_3 \\ U_R &= \beta_1 S_1 + \beta_2 S_2 + \beta_3 S_3 \end{aligned}$$

As a rule of thumb, we have $m - 1$ intermediate states.

As a rule of thumb, we have $m-1$ intermediate states between m discontinuities. The jump along the p-th discontinuity is

$$[u]_p = (\beta_p - \alpha_p)\, S_p \qquad 1 \leq p \leq m$$

$$\begin{aligned}[f]_p &= \underline{\underline{A}} \cdot [u]_p \\ &= (\beta_p - \alpha_p)\, \underline{\underline{A}} \cdot \underline{S}_p \\ &= (\beta_p - \alpha_p)\, \lambda_p\, \underline{S}_p = \lambda_p [u]_p\end{aligned}$$

Thus, RH condition : $[f]_p = \lambda_p [u]_p$

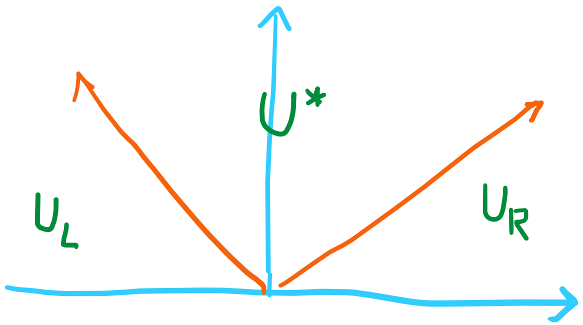
We see that

$$
\begin{aligned}
\underline{U}_R - \underline{U}_L &= \underbrace{\underline{U}_R - \underline{U}^{*}}_{[U]_m} + \underbrace{\underline{U}^{*} - \underline{U}^{**}}_{[U]_{m-1}} + \underline{U}^{**} + \cdots - \underline{U}_L \\
&= [U]_1 + [U]_2 + \cdots + [U]_m \\
&= \sum_{p=1}^{m} [U]_p \\
&= \sum_{p=1}^{m} (\beta_p - \alpha_p)\, \underline{S}_p = \underline{\underline{S}}\,(\beta - \alpha)
\end{aligned}
$$

This describes the solution of the Riemann problem:

1. Find the eigenvalues and eigenvectors of the system matrix
2. Assemble the intermediate states by the procedure above

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \qquad \lambda_1 = 1, \qquad \lambda_2 = -1, \qquad S_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \qquad S_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$



We seek U^* such that

$$A(U^* - U_L) = (-1)(U^* - U_L) \qquad 1)$$

$$A(U_R - U^*) = (1)(U_R - U^*) \qquad 2)$$

Two equations with two unknowns.

Explicitly

1) + 2) $\Rightarrow$ $$A(U_R - U_L) = U_R + U_L - 2U^*$$

$$U^* = \frac{1}{2}(U_L + U_R) - \frac{1}{2}A(U_R - U_L)$$

We have discussed linear systems with detail now. What about nonlinear systems?

$$\partial_t U + \partial_x \left[f(U) \right] = 0$$

$$\partial_t U + \underbrace{\nabla_x f(U)}_{A(U)} \cdot \partial_x U = 0$$

$$\partial_t U + A(U) \cdot \partial_x U = 0$$

We call $A(U) = \nabla_x f(U)$ the <u>flux Jacobian</u>

More details on nonlinear systems soon…