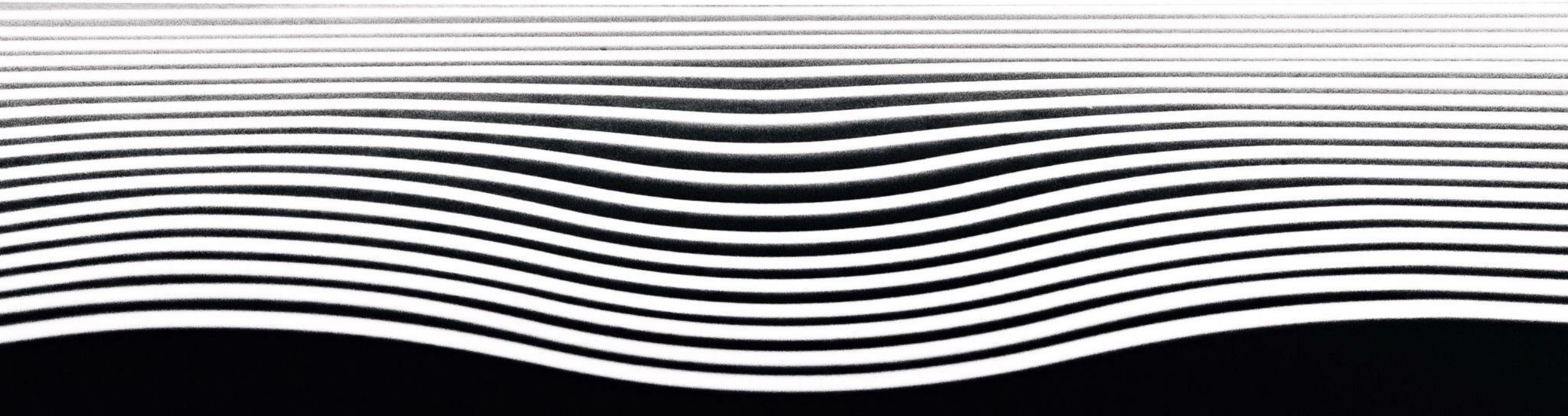
Numerical methods for conservation laws
10: Finite Volume Schemes



- We have discussed finite-difference schemes
- Next, we discuss finite volume methods

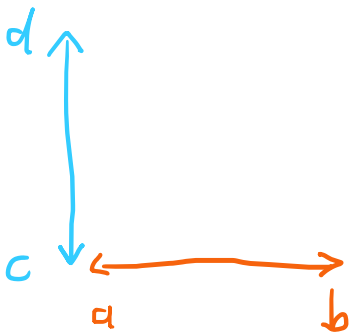
Finite volume methods are another large class of numerical schemes for conservation laws.

The conservation law $u_t + f(u)_x = 0$ is written in integral form: whenever

$$\omega = [a, b] \times [c, d] \subseteq \mathbb{R} \times \mathbb{R}_0^\infty$$

is a small subset of spacetime, we have

$$\iint_\omega u_t = -\iint_\omega f(u)_x$$



We integrate in time and space, respectively, which leads to

$$\int_a^b u(x, d) - u(x, c)\, dx = -\int_c^d f\big(u(b, t)\big) - f\big(u(a, t)\big)\, dt$$

Suppose we have a discretization of spacetime defined by nodal points in space and time steps. Consider

$$\int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} u(x,t^{n+1}) - u(x,t^{n})\,dx = -\int_{t^n}^{t^{n+1}} f\left(u\left(x_{i+\frac{1}{2}},t\right)\right) - f\left(u\left(x_{i-\frac{1}{2}},t\right)\right)\,dt$$

We define cell averages and local fluxes:

$$\overline{U}_i^n = \frac{1}{h}\int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} u(x,t^n)\,dx\,, \qquad \overline{F}_{i+\frac{1}{2}}^{n} = \frac{1}{k}\int_{t^n}^{t^{n+1}} f\left(u\left(x_{i+\frac{1}{2}},t\right)\right)\,dt$$

Hence, at all times we have the identity:

$$\overline{U}_i^{n+1} = \overline{U}_i^n - \frac{k}{h}\left(\overline{F}_{i+\frac{1}{2}}^{n} - \overline{F}_{i-\frac{1}{2}}^{n}\right)$$

This is the prototype for a numerical method. Instead of approximating point values, we approximate cell averages.

We haven't made any numerical approximation yet. The above formula will be used as a blueprint to build numerical methods. The key idea is that we approximate the integrals.

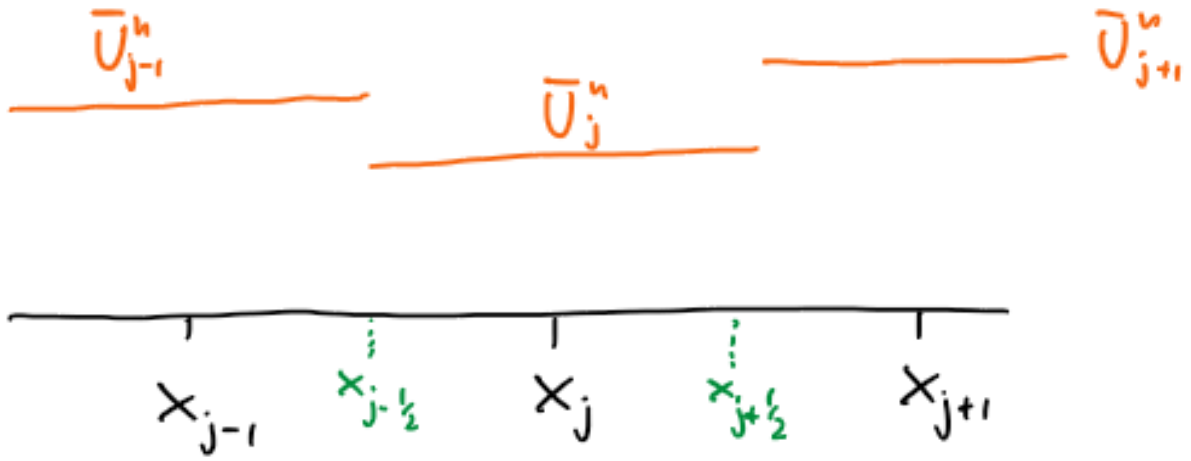
- Only aim to compute the averages $\overline{U}_i^n$, not the underlying function
- Only approximate the time integral of the fluxes.

This leads to a large class of numerical methods, known as finite volume methods.

Gudonov scheme:
At each time step, we have a piecewise constant function, constant over each cell.
Near the cell interfaces, we have local Riemann problems.

We evolve the system for a short time interval. Then the rarefaction and shockwaves emerging from the interfaces will not interact with each other, provided that the time interval is short enough.

We have enough data to exactly compute the fluxes, at least in principle!
In practice, we still need approximations.



We have the exact formula:

$$\bar{U}_i^{n+1} = \bar{U}_i^n - \frac{k}{h}\left(\bar{F}^n_{i+\frac{1}{2}} - \bar{F}^n_{i-\frac{1}{2}}\right)$$

The averages and fluxes still need to be computed. Several different approximations are possible.

1 Given $\bar{U}^n_j$, we define a polynomial of degree $p+q$ by setting

$$\frac{1}{h}\int_{x_{\ell-1/2}}^{x_{\ell+1/2}} p_j(x)\,dx = \bar{U}^n_\ell, \qquad \ell = j-p, \dots, j+q$$

Most simple choice: $p = q = 0$, $p_j = U^n_j$

2 Locally at the interface $x_{j-\frac{1}{2}}$, we study the local Riemann problem

$$\partial_t u + \partial_x f(u) = 0$$

$$u_0(x) = \begin{cases} p_{j-1}(x_{j-\frac{1}{2}}) & \text{if } x < x_{j-\frac{1}{2}} \\ p_j(x_{j-\frac{1}{2}}) & \text{if } x > x_{j-\frac{1}{2}} \end{cases}$$

3 Whatever emerges at the interface will not reach the neighboring interfaces if

$$|f'| < \frac{h}{k} \qquad \frac{k}{h}|f'| < 1$$

4 Given that local solution, we <u>exactly</u> compute

$$\overline{F}^{n}_{j-\frac{1}{2}} = \int_{t^n}^{t^{n+1}} f\left(u(x_{j-\frac{1}{2}}, t) \right) dt$$

Problem: the integral cannot be computed in practice

Compromise: solve and integrate approximately

Ex 1 Transport eqv: $u_t + c u_x = 0$

Local problem has explicit solution

Ex 2 Burgers' equation

Local problem has explicit solution:

a) Shockwave $u_l > u_r$

b) Rarefaction wave $u_l < u_r$