Now let us discuss implementation aspects of DG schemes

The theoretical idea is similar to finite element methods

We use a map from a reference cell into the physical cell

$$\chi_j : [-1, 1] \longrightarrow [x_{j-\frac{1}{2}}, x_{j+\frac{1}{2}}]$$

$[-1, 1]$: reference cell in 1D; $[x_{j-\frac{1}{2}}, x_{j+\frac{1}{2}}]$: j-th cell

$$r \longmapsto x_{j-\frac{1}{2}} + \frac{1+r}{2} h_j$$

where $h_j = x_{j+\frac{1}{2}} - x_{j-\frac{1}{2}}$. With that we have on the j-th cell

$$\phi_j^i(\chi_j(r)) = \phi^i(r)$$

ϕ_j^i: i-th basis function on the j-th cell; ϕ^i: i-th basis function on the reference cell

$$\left[\phi_j^i(x) = \phi^i(\chi_j^{-1}(x)) \right]$$

and

$$\begin{aligned} U_h^j(x,t) &:= U_h^j(\chi(r), t) \\ &= \sum_{i=0}^{m} U_j^i(t)\ \phi_j^i(\chi(r)) \\ &= \sum_{i=0}^{m} U_j^i(t)\ \phi^i(r) \end{aligned}$$

One can show that

$$M^j = \frac{h_j}{2} M, \qquad S^j = S$$

M^j: j-th physical mass matrix; M: ref mass matrix; S^j: j-th physical stiffness matrix; S: ref stiffness matrix

Thus we can transform the computations to the ref. cell.

Consider the approximation

$$U_h(r) := \sum_{i=0}^{m} \hat{U}^i \phi^i(r)$$

$\hat{U}^i$: module coefficients

This is the module representation. What is a good choice of the basis on the reference cell?

(A) $$\phi^i(r) := r^i \qquad \text{(monomials)}$$

The L^2 projection U_h of U onto the approximation space is calculated via a LSE with the local mass matrix.

$$\int_{-1}^{1} U_h(r)\, \phi^l(r)\, dr \equiv \sum_{i=0}^{m} \hat{U}^i \int_{-1}^{1} \phi^i(r)\, \phi^l(r)\, dr = \int_{-1}^{1} U(r)\, \phi^l(r)\, dr$$

Solving a system with the mass matrix is "difficult" if we choose the monomial basis. Then the mass matrix has entries resembling a Hilbert matrix:

$$M_{kl} = \frac{1 + (-1)^{k+l}}{k+l+1}$$

But such matrices are ill-conditioned for larger m.

(B) Alternatively we can choose an orthonormal basis, constructed via Gram-Schmidt. Then the mass matrix is the identity matrix.

Applying Gram-Schmidt to the monomials produces the Legendre polynomials:

$$\phi^0(r) = \frac{1}{\sqrt{2}}, \qquad \phi^1(r) = \sqrt{\frac{3}{2}}\, r,$$

$$r\phi^i = \alpha_i \phi^{i-1} + \alpha_{i+1} \phi^{i+1} \qquad \text{(recurrence relation)}$$

$$r\phi^i = \alpha_i \phi^{i-1} + \alpha_{i+1} \phi^{i+1}$$ (recurrence relation)

We still want to calculate the integrals

$$\int_{-1}^{1} U(r)\, \phi^l(r)\, dr$$

We can use Gauss-Legendre quadrature:

$$\int_{-1}^{1} U(r)\, \phi^l(r)\, dr := \sum_{k=0}^{m} U(r_k)\, \phi^i(r_k)\, \omega_k$$

r_k: quadrature nodes; ω_k: quadrature weights

Exact for polynomials up to degree $2m+1$.

Alternatively, we use classical interpolation to construct U_h.

Find U_h s.t. $U(r_l) = U_h(r_l)$

at distinct nodal points $r_0, r_1, \ldots, r_m \in [-1, 1]$

$$U(r_l) = \sum_{i=0}^{m} \hat{U}^i \phi^i(r_l) \qquad l = 0, \ldots, m$$

Another linear system of equations. The corresponding matrix is the Vandermonde matrix

$$V_{lk} = \phi^k(r_l)$$

The nodal representation is the representation

$$U_h(r) = \sum_{i=0}^{m} U(r_i)\, l^i(r)$$

$$U_h(r) = \sum_{i=0}^{m} U(r_i)\, \ell^i(r)$$

$U(r_i)$: nodal coefficients; $\ell^i(r)$: Lagrange polynomial

The transformation between nodal and modal coefficients is just a linear transformation.

The two representations are generally not the same.

$$U_h(r) := \sum_{i=0}^{m} \hat{U}^i\, \phi^i(r) \qquad U_h(r) := \sum_{i=0}^{m} \tilde{U}^i\, \phi^i(r)$$

$\hat{U}^i$: via L^2 projection — requires integrals of U

$\tilde{U}^i$: via interpolation — requires point values of U

Major application of the L^2 projection and nodal interpolation:

Polynomial approximation of non-polynomial terms for the computation of the matrices in the discrete formulation (or RHS)