30 Discontinuous Galerkin methods 3

We have so far analyzed the stability results of DG schemes. What can we say about accuracy?

Our error analysis will mimic the techniques for FD/FV methods. We consider the linear scalar problem

$$U_t + LU = 0$$

We define the error as

$$\varepsilon_h(x,t) := U(x,t) - U_h(x,t)$$

Then

$$\partial_t \varepsilon_h + LU - L_h U_h = 0$$

$$\Leftrightarrow \quad \partial_t \varepsilon_h + L_h U - L_h U_h = L_h U - LU$$

$$\Leftrightarrow \quad \partial_t \varepsilon_h + L_h \varepsilon_h = (L_h - L)U =: T(U)$$

truncation error

Thus

$$\varepsilon_h(x,t) = \exp(-L_h t)\, \varepsilon_h(x,0) + \int_0^t \exp(L_h(s-t))\, T(U)\, ds$$

We define the space norm

$$\| U(t) \|_h^2 = \sum_{j=1}^{N} \| U(t) \|_{L^2(D_j)}^2$$

With that definition in mind

$$\| \mathcal{E}_h(t) \|_h \leq \| \exp(-L_h \cdot t) \|_h \cdot \| \mathcal{E}_h(0) \|_h$$

$$+ \int_0^t \| \exp(L_h (t-s)) \|_h \cdot \| T(U(s)) \|_h \, ds$$

For convergence we need

i) $\| \mathcal{E}(0) \|_h \longrightarrow 0$

ii) $\| T(U(s)) \|_h \longrightarrow 0$

(consistency)

iii) $\| \exp(L_h (s-t)) \|_h \leq C$ (stability)

The error is the sum of two parts:

1) Approximation of the initial data:

$$\| \mathcal{E}(0) \|_h \leq c \, h^{m+1}$$

ii) Accumulation of the truncation error:

more complicated, see the book.

One can show that

$\boxed{1D}$ $\quad \| u - u_h \|_h \leq C h^{m+1/2}$ (sub-optimal)

$\| u - u_h \|_h \leq C h^{m+1}$ for linear problems with strict upwinding

$\boxed{2D}$ $\boxed{3D}$ $\quad \| u - u_h \|_h \leq C h^{m+1/2}$ (sub-optimal)