

Suppose we pursue solving the conservation law:

$$\partial_t u + \underset{\text{div}}{\partial_x F(u)} = 0$$

We proceed in the spirit of Galerkin methods, with modifications.

We first develop a semi-discrete scheme, not discretizing in time.

① We decompose the domain Ω into non-overlapping cells

$$\Omega = \bigcup_{j=1}^N D_j$$

Example:

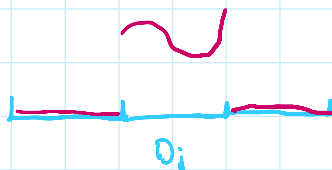


② In each cell D_j , we define a local basis of the space of polynomials $V_h^j = P_m$ of degree m over D_j .

$$\left\{ \Phi_j^i \right\}_{i=0}^m \quad m+1 \text{ functions}$$

These functions vanish outside of D_j

Example

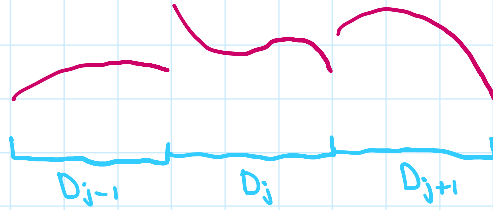


③ Define the space of functions that are piecewise polynomial over the mesh

$$V_h := \bigoplus_{j=1}^N V_h^j$$

This also known as the space of broken polynomials
 The functions V_h may be discontinuous across cell boundaries

Example



④

On each cell we define a local solution

$$U_h^j(x, t) = \underbrace{\sum_{i=0}^m U_j^i(t) \Phi_j^i(x)}_{\in V_h^j} \quad x \in D_j$$

Their sum is the global function

$$U_h(x, t) = \sum_{j=1}^N U_h^j(x, t)$$

⑤

We define the method by a Galerkin formulation

For each cell D_j we require

$$\int_{D_j} \partial_t U_h^j v_h + \operatorname{div} F(U_h^j) v_h \, dx = 0$$

for all $v_h \in V_h^j$. Equivalently, for all $0 \leq i \leq m$

$$\int_{D_j} \partial_t U_h^j \Phi_j^i + \operatorname{div} F(U_h^j) \Phi_j^i \, dx = 0$$

We perform an integration by parts in space

$$\int_{D_j} \partial_t U_h^j \Phi_j^i \, dx - \int_{D_j} F(U_h^j) \nabla \Phi_j^i \, dx = - \oint_{\partial D_j} \Phi_j^i F(U_h^j) \cdot \vec{n} \, ds$$

outward normal
with unit length

Note that these are separate formulations for each single cell

Note that these are separate formulations for each single cell
 We need to couple the different cells.

Idea: we replace the boundary integral on the right-hand side
 by a numerical flux. At each interface of the cell D_j we
 incorporate data from the neighbouring cell.

Remark: consider the test function $v_h = \frac{1}{h}$ constant

$$\partial_t \int_{D_j} u_h^i \cdot \frac{1}{h} dx = - \frac{1}{h} \oint_{\partial D_j} F(u_h^i) \cdot \vec{n} ds$$

↑ time-derivative of the average

Local conservation of the "mass" over each cell

For simplicity we consider a one-dimensional setting.

$$D_j = [x_{j-\frac{1}{2}}, x_{j+\frac{1}{2}}]$$

The formulation reads:

$$\int_{D_j} \partial_t u_h^i \Phi_j^i dx - \int_{D_j} F(u_h^i) \nabla \Phi_j^i dx = - \Phi_j^i F(u_h^i) \Big|_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}}$$

SS

We introduce a numerical flux F^*
 which provides the coupling between the
 different cells.

$$- \Phi_j^i F^* \Big|_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}}$$

This is called the weak formulation of the DG method.

We integrate the left-hand side in space again.

$$\int_{D_j} \partial_t u_h^i \Phi_j^i + \operatorname{div} F(u_h^i) \Phi_j^i dx = - \Phi_j^i (F(u_h^i) - F^*) \Big|_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}}$$

This is called the strong formulation of the DG method

Each formulation has benefits in the theoretical analysis.

We define a few matrices that will be important in the discussion

$$M_{kl}^j = \int_{D_j} \Phi_j^k \Phi_j^l dx, \quad S_{kl}^j = \int_{D_j} \Phi_j^k \partial_x \Phi_j^l dx$$

↑
"mass matrix", symmetric

We approximate $F(U_h^j)$ by its projection onto the local space over D_j

$$F_h^j(t, x) = \sum_{i=0}^m F_j^i(t) \Phi_j^i(x), \quad x \in D_j$$

such that at time t

$$\int F_h^j(t, x) \Phi_j^i(x) dx = \int F(U_h^j(t)) \Phi_j^i(x) dx$$

This uses the Gram matrix of the basis Φ_j^i over D_j
(mass matrix)

With those matrices we write the weak form in terms of coefficients

$$M^j \partial_t U^j + (S^j)^T F^j = - \Phi^j F^* \Big|_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}}$$

where

$$U^j = [U_j^0, \dots, U_j^m]^T$$

$$F^j = [F_j^0, \dots, F_j^m]^T$$

$$\Phi^j = (\Phi_j^0, \dots, \Phi_j^m)^T \text{ at } x_{j \pm \frac{1}{2}}$$

$$\left(- \Phi_j^i F^* \Big|_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} \right)_{i=0}^m$$

Similarly, the strong form reads in terms of coefficients

$$M^j \partial_t U^j + S^j F^j = - \mathbb{I}^j (F - F^*) \Big|_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}}$$

So what should we pick as the numerical flux?
We search

$$F^* = F^*(U^-, U^+)$$

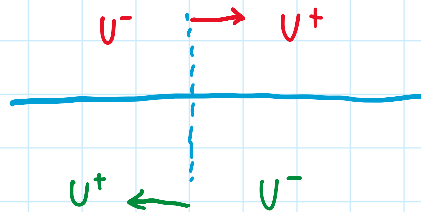
such that

- F^* is monotone : $F^*(\uparrow, \downarrow)$
- consistent : $F^*(U, U) = F(U)$
- single-valued : $F^*(U, V) = F^*(V, U)$

Example : Lax-Friedrichs

$$F^*(U^-, U^+) = \frac{F(U^-) + F(U^+)}{2} - \frac{\alpha}{2} \hat{n} (U^+ - U^-)$$

$\hat{n}$ outward pointing, normalized ± 1



Theoretical properties :

Consider the weak form

$$M^j \partial_t U^j - (S^j)^T F^j = - \Phi^j F^* \Big|_{x_{j-1/2}}^{x_{j+1/2}}$$

Consider any smooth test function $\psi(x, t)$ and project it onto the global space V_h :

$$\psi(x, t) \approx \psi_h(x, t) = \sum_{j=1}^N \psi_h^j(x, t)$$

where

$$\int \psi_h^j(x, t) \Phi_j^i(x) dx = \int \psi(x, t) \Phi_j^i(x) dx$$

Let $\psi^j = (\psi_j^0, \dots, \psi_j^m)$ be the local coefficients in the j -th cell. Then in that cell:

$$(\psi^j)^T M^j \partial_t U^j - (\psi^j)^T (S^j)^T F^j = - (\psi^j)^T (\Phi^j F^*) \Big|_{x_{j-1/2}}^{x_{j+1/2}}$$

$$\begin{array}{ccc} \uparrow & & \uparrow \\ \int_{D_j} \psi_h^j M^j \partial_t U^j dx & & \int_{D_j} \partial_x \psi_h^j \cdot F_h^j dx \end{array}$$

① Assume $\psi(x, t) = \frac{1}{h}$, so that $\psi_h^j = \frac{1}{h}$. Then

$$\partial_t \underbrace{\frac{1}{h} \int_{D_j} U_h^j dx}_{\text{Avg over } D_j = \bar{U}_h^j} = -\frac{1}{h} (F_{j+1/2}^* - F_{j-1/2}^*)$$

This is the basic form of a FV scheme.

Summing up:

$$\begin{aligned}\partial_t \sum \bar{U}_h^j &= -\frac{1}{h} \sum_j (F_{j+\frac{1}{2}}^* - F_{j-\frac{1}{2}}^*) \\ &= -\frac{1}{h} (F_{N+\frac{1}{2}}^* - F_{0-\frac{1}{2}}^*)\end{aligned}$$

From local conservation we get global conservation.

② Let us assume that ψ is compactly supported on Ω and vanishes for large t .

We integrate over the time interval and integrate by parts in time. This leads to

$$\begin{aligned}& - \int_0^T \int_{D_j} \underline{U_h^j \partial_t \psi_h^j} + F_h^j \nabla \psi_h^j \, dx \, dt \\ & + \int_{D_j} \psi_h^j(x, 0) U_h^j(x, 0) \, dx \\ & = - \int_0^T \psi_h^j F^* \Big|_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} \, dt\end{aligned}$$

over each cell. We study the right-hand side

$$- \int_0^T \psi_h^j F^* \Big|_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} \, dt$$

As h goes to zero or m increases, we have

$$\psi_h^j \longrightarrow \psi$$

in all relevant metrics. But then

$$\sum_{j=1}^N \int_0^T \psi_h^j F^* \Big|_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} \, dt \longrightarrow \int_0^T \psi F^* \Big|_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} \, dt$$

$$\sum_{j=1}^N \int_0^T \psi_h^j F^* \Big|_{x_{j-1/2}}^{x_{j+1/2}} dt \longrightarrow 0.$$

If U_h converges to some U in the meanwhile, then

$$- \sum_j \int_0^T \int_{D_j} U_h^j \partial_t \psi_h^j + F_h^j \nabla \psi_h^j dx dt$$

$$+ \sum_j \int_{D_j} \psi_h^j(x, 0) U_h^j(x, 0) dx$$

$$= - \int_0^T \int_{\Omega} U_h \partial_t \psi_h + F_h \overset{\text{piecewise}}{\nabla} \psi_h dx dt + \int_{\Omega} \psi_h(x, 0) U_h(x, 0) dx$$

converges to

$$- \int_0^T \int_{\Omega} U \partial_t \psi + F(U) \nabla \psi dx dt + \int_{\Omega} \psi(x, 0) U(x, 0) dx$$

which is the weak formulation of the conservation law.

That is if U is the limit, then it is a weak solution.

③ Consider the strong discrete formulation:

$$M^j \partial_t U^j + S^j F^j = \Phi^j(F - F^*) \Big|_{x_{j-1/2}}^{x_{j+1/2}}$$

We take the scalar product with U^j :

$$\langle U^j, M^j \partial_t U^j \rangle + \langle U^j, S^j F^j \rangle = U_h^j (F - F^*) \Big|_{x_{j-1/2}}^{x_{j+1/2}}$$

That is,

$$\frac{1}{2} \partial_t \|U_h^j(t)\|_{L^2(D_j)}^2 + \int_{D_j} U_h^j \nabla F_h^j dx = U_h^j (F - F^*) \Big|_{x_{j-1/2}}^{x_{j+1/2}}$$

$$\frac{1}{2} \partial_t \|U_h^j(t)\|_{L^2(D_j)}^2 + \int_{D_j} U_h^j \nabla F_h^j dx = U_h^j (F - F^*) \Big|_{x_{j-1/2}}^{x_{j+1/2}}$$

Let us assume that we have the identity

$$F(U_h^j(x,t)) = F_h^j(x,t)$$

For example, that is true if F is a polynomial in U .
(e.g. Burgers' equation)

This assumption implies that

$$\begin{aligned} \int_{D_j} U_h^j \nabla F_h^j dx &= \int_{D_j} \underbrace{U_h^j DF(U_h^j)}_{D \tilde{F}(U_h^j)} \nabla U_h^j dx \\ &= \int_{D_j} D \tilde{F}(U_h^j) \cdot \nabla U_h^j dx \\ &= \int \operatorname{div} \tilde{F}(U_h^j) dx = \tilde{F} \Big|_{x_{j-1/2}}^{x_{j+1/2}} \end{aligned}$$

Hence we find that

$$\frac{1}{2} \partial_t \|U_h^j\|_{L^2(D_j)}^2 = \left(-\tilde{F} + U_h^j (F - F^*) \right) \Big|_{x_{j-1/2}}^{x_{j+1/2}}$$

Summing over all j leads to

$$\frac{1}{2} \partial_t \|U_h^j\|_{L^2(\Omega)}^2 = \sum_j \left(-\tilde{F} + U_h^j (F - F^*) \right) \Big|_{x_{j-1/2}}^{x_{j+1/2}}$$

We further analyze the right-hand side. At the interface $x_{j+1/2}$ we observe

$$\begin{array}{c} U^- \quad | \quad U^+ \\ \hline D_j \quad x_{j+1/2} \quad D_{j+1} \end{array}$$

$$\overline{D_j \quad x_{j+1/2} \quad D_{j+1}}$$

$$F^- := -\tilde{F}(U^-) + U^-(F(U^-) - F^*)$$

$$F^+ := \tilde{F}(U^+) - U^+(F(U^+) - F^*)$$

At the interface

$$\begin{aligned} F^- + F^+ &= -\tilde{F}(U^-) + U^-(F(U^-) - F^*) + \tilde{F}(U^+) - U^+(F(U^+) - F^*) \\ &= -\tilde{F}(U^-) + U^-F(U^-) + \tilde{F}(U^+) - U^+F(U^+) - F^*(U^- - U^+) \end{aligned}$$

If this interface value is non-positive, then the energy is non-increasing:

$$\partial_t \|U_h\|_{L^2(\Omega)}^2 \leq 0$$

When is that the case?

Define $G(U) = U F(U) - \tilde{F}(U)$. Then

$$G'(U) = F(U) + \underbrace{U F'(U) - \tilde{F}'(U)}_{=0} = F(U)$$

By the mean value theorem, there exists ξ between U^+ and U^- s.t.

$$\begin{aligned} G(U^+) - G(U^-) &= G'(\xi) [U^+ - U^-] \\ &= F(\xi) [U^+ - U^-] \end{aligned}$$

We want

$$\begin{aligned} G(U^-) - G(U^+) - F^* \cdot (U^- - U^+) &\leq 0 \\ \Leftrightarrow -F(\xi)(U^+ - U^-) - F^* \cdot (U^- - U^+) &\leq 0 \\ \Leftrightarrow (F(\xi) - F^*)(U^- - U^+) &\geq 0 \end{aligned}$$

$\Leftrightarrow$

$$(F'(\xi) - F^*)(U^- - U^+) \leq 0$$

This last condition is known as E-Flux condition

What kind of fluxes satisfy this E-Flux condition?

Suppose that $F^* = F^*(U^-, U^+)$ is monotone and consistent.

Then $F(\xi) = F^*(\xi, \xi)$.

Let $s, t \in [0, 1]$ and suppose that

$$\xi = U^+ + s(U^- - U^+) = U^- + t(U^+ - U^-)$$

Via Taylor expansion

$$\begin{aligned} F^*(\xi, \xi) &= F^*(U^-, U^+) + s(U^- - U^+) \partial_2 F^*(U^-, U^+) \\ &\quad + t(U^+ - U^-) \partial_1 F^*(U^-, U^+) \end{aligned}$$

Hence

$$\begin{aligned} &[F(\xi) - F^*(U^-, U^+)](U^- - U^+) \\ &= \underbrace{s}_{\geq 0} \underbrace{(U^- - U^+)^2}_{\geq 0} \underbrace{\partial_2 F^*(U^-, U^+)}_{\leq 0} - \underbrace{t}_{\geq 0} \underbrace{(U^- - U^+)^2}_{\geq 0} \underbrace{\partial_1 F^*(U^-, U^+)}_{\geq 0} \end{aligned}$$

The right-hand side is non-positive.

That is, monotone fluxes are E-Fluxes.

Thus, for a polynomial flux F that is monotone and single-valued, we have L^2 -stability of the solution.