

To appreciate discontinuous Galerkin schemes, we first discuss continuous Galerkin schemes.

We recall that conservation laws are a limit of vanishing viscosity of PDEs of the following form:

$$\partial_t u + \partial_x F(u) = \varepsilon \partial_{xx} u$$

We multiply by a function v , and integrate in space

$$\int \partial_t u \cdot v \, dx + \int \partial_x F(u) \cdot v \, dx = \varepsilon \int \partial_{xx} u \cdot v \, dx$$

Suppose that u has vanishing boundary values, then integration by parts in space gives

(weak formulation)

$$\int \partial_t u \cdot v \, dx - \int F(u) \cdot \partial_x v \, dx = -\varepsilon \int \partial_x u \cdot \partial_x v \, dx$$

We may search the solution at each time t in the Sobolev space H^1 .

We recall L^2 , the space of functions that are square-integrable.

$$\int u^2 \, dx < \infty$$

A function $u \in L^2$ has got a weak derivative $w \in L^2$ if for all compactly supported smooth functions φ :

$$\int u \cdot \partial_x \varphi \, dx = - \int w \cdot \varphi \, dx$$

We define H^1 as the subspace of functions in L^2 that have a weak derivative in L^2 .

We seek a solution as a function

$$u: [0, T] \rightarrow H^1$$

such that for all $t \in [0, T]$ and all $v \in H^1$:

$$\int \partial_t u(t) v \, dx = -\varepsilon \int \partial_x u(t) \partial_x v \, dx + \int F(u(t)) \partial_x v \, dx$$

Can we develop a numerical method from here?

$\Rightarrow$ continuous Galerkin approach

Consider a stationary problem where the time-derivative is zero

$$\varepsilon \partial_{xx} u - \partial_x F(u) = f$$

over some domain Ω . Suppose that homogeneous Dirichlet BC are imposed:

$$u(x) = 0 \quad \text{along } \partial\Omega$$

We formulate a weak formulation via integration against a test function and integration by parts

$$\varepsilon \int \partial_x u \partial_x v \, dx - \int F(u) \partial_x v \, dx = \int f v \, dx$$

We seek the (stationary) solution u in the space

$$H_0^1(\Omega) = \left\{ u \in H^1(\Omega) \mid u(x) = 0 \text{ along } \partial\Omega \right\}$$

and test it against functions $v \in H_0^1(\Omega)$ from the same space.

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not discussed here - "trace of Sobolev functions"

Example: if $F(u) = -cu$ then we get

$$\varepsilon \int \partial_x u \cdot \partial_x v \, dx + \int c u v \, dx = \int f v \, dx$$

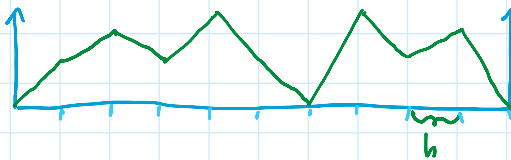
When $\varepsilon \ll 1$ is very small but $c \neq 0$,
then this is computationally hard!

Galerkin approach: we solve the weak formulation
but we search u_h in a subspace $V_h \subseteq H_0^1(\Omega)$
and test against functions v_h from the same subspace.

$$\varepsilon \int \partial_x u_h \partial_x v_h \, dx - \int F(u_h) \partial_x v_h \, dx = \int f v_h \, dx$$

What is a reasonable subspace V_h ?

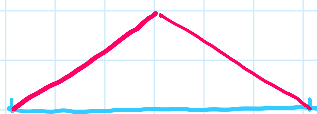
We can use the Lagrange space of continuous piecewise linear functions
with respect to some mesh



This is a subspace of H_0^1 . For every $v_h \in V_h$, we have that

i) $v_h \in L^2(\Omega)$

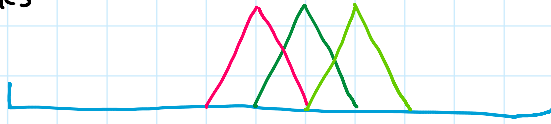
ii) The spatial derivative is piecewise constant



weak derivative
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A basis of that vector space is given by the hat functions  $\Phi_i$  over the inner nodes



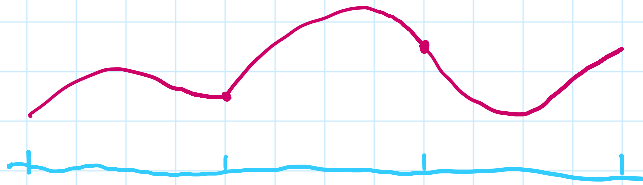
We can write every  $v_h \in V_h$  as a linear combination / superposition of those hat functions

$$v_h = \sum_i v_i \Phi_i$$

↖ hat function

↖ coefficient

Remark: More generally, we can pick  $V_h$  as the space of continuous functions that piecewise polynomial with some polynomial degree  $m$ .



Example (continued): consider the linear problem

$$\varepsilon \int \partial_x u_h \cdot \partial_x v_h \, dx + \int c u_h v_h \, dx = \int f v_h \, dx$$

It suffices to test only against hat functions since every other test function is a linear combination of those. We expand

$$u_h = \sum_i u_i \Phi_i$$

and thus get for each index  $j$ :

$$\varepsilon \int \sum_i u_i \partial_x \Phi_i \partial_x \Phi_j + \int \sum_i u_i c \Phi_i \Phi_j = \int f \Phi_j \, dx$$



We rearrange this and recognize a linear system of equations:  
for each  $j$ :

$$\sum_i u_i \cdot \left[ \varepsilon \int \partial_x \Phi_i \partial_x \Phi_j + \int c \Phi_i \Phi_j \right] = \int f \Phi_j dx$$

Thus finding  $u_h$  amounts to solving a linear system of equations to get the coefficients  $u_i$ .

Do the weak formulation and the Galerkin approximation have unique solutions, and if yes, how good is the approximation?

Generally yes, and the solutions satisfy

$$\|u - u_h\|_{H^1} \leq C \inf_{v_h \in V_h} \|u - v_h\|_{H^1}$$

where

$$\|v\|_{H^1} := \|v\|_{L^2} + \|\partial_x v\|_{L^2}$$

independent of  $u$  and  $f$  and  $h$ ,  
only depends on PDE coeff.

That is, the Galerkin method is as good as the best approximation within  $V_h$  up to a constant  $C$

One can show (Bramble-Hilbert lemma)

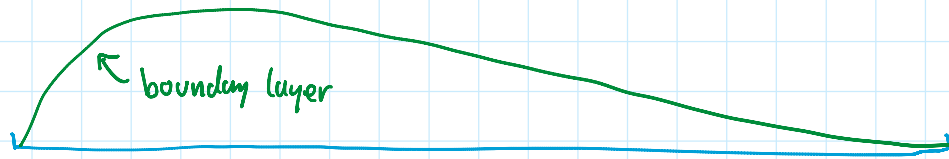
$$\inf_{v_h \in V_h} \|u - v_h\|_{H^1} \lesssim$$

$$\underline{h} \|u\|_{H^2}$$

$L^2$ -norms of  
 $u, \partial_x u, \partial_{xx}^2 u$

Problem: That constant  $C$  might be quite bad

Example (continued): When  $\epsilon$  is very small in comparison to  $c$ , then the solution will develop steep gradients at the boundary

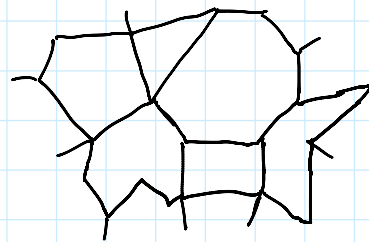
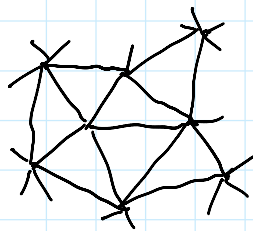
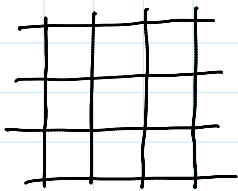


One needs small  $h$  to get good approximations

### Conclusion for conservation laws:

When the solution to a conservation law develops steep gradients and jumps, then continuous Galerkin methods will struggle to approximate the solution

However, Galerkin methods are well-understood from a theoretical POV and easily applied to complicated geometries in any dimension



Can we leverage that?