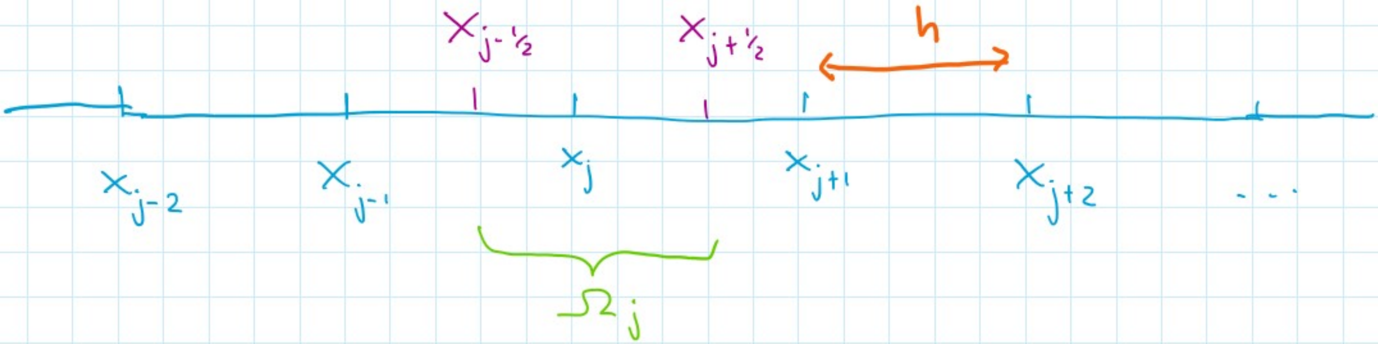
11. 1-3



Suppose we have a function u that is sufficiently smooth and only know its cell averages

$$\bar{u}_j = \frac{1}{h} \int_{\Omega_j} u(x)\, dx$$

Can we reconstruct the values of u at the interfaces (approximately)?

Suppose we have cell averages

$$\bar{u}_{j-p}, \ \dots, \ \bar{u}_{j+q}$$

We seek a polynomial π of order $m-1$, where $m = p+q+1$ is the number of points, such that

$$u(x_{j+\frac{1}{2}}) = \pi(x_{j+\frac{1}{2}}) + O(h^m)$$

and such that

$$\frac{1}{h}\int_{\Omega_i}\pi(x)\,dx = \bar{u}_i, \qquad i = j-p,\dots,j+q$$

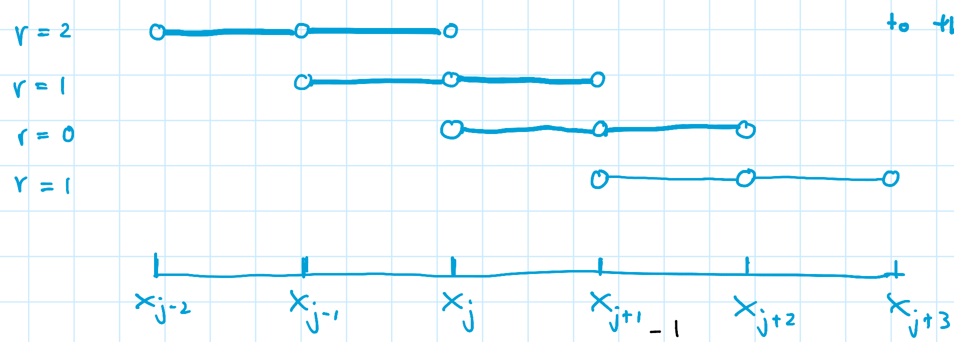
(m constraints)

We have numerous "stencils" of interpolation points that differ by the shift parameter.

$$j-r,\ j-r+1,\ \dots,\ j-r+m$$

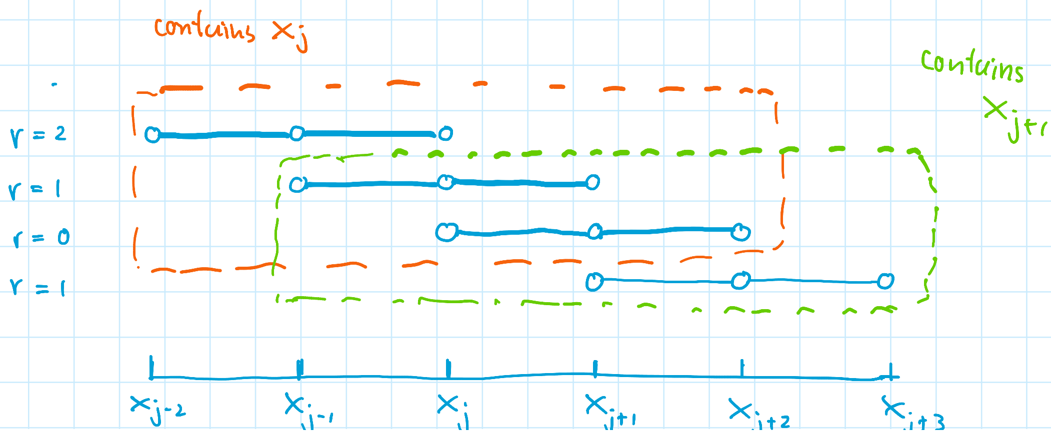
$\boxed{\text{Example}}$ $m = 3$

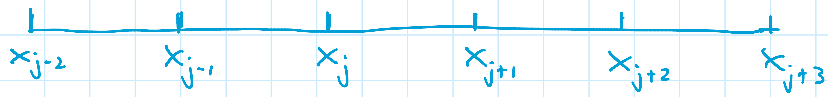
r = shift parameter to the left



In order to contain x_j or x_{j+1} in the stencil, we must have the shift

$$-1 \le r \le m$$





Thus we have two families of stencils of size m

Consider an antiderivative V of u :

$$V'(x) = u(x)$$

For a given shift $-1 \leq r \leq m$ we contruct a polynomial $\Pi(x)$ of degree m which interpolates V at the cell interfaces

$$\Pi(x_{j-r+i-\frac{1}{2}}) = V(x_{j-r+i-\frac{1}{2}}), \quad i = 0, \dots, m$$

($m+1$ constraints)

We consider its derivative

$$\pi(x) = \partial_x \Pi(x)$$

We calculate the integral of π over cells. For $\ell = j-r, \dots, j-r+m$ we have

$$\frac{1}{h}\int_{x_{\ell-\frac{1}{2}}}^{x_{\ell+\frac{1}{2}}} \pi(\xi)\, d\xi = \frac{1}{h}\int_{x_{\ell-\frac{1}{2}}}^{x_{\ell+\frac{1}{2}}} \Pi'(\xi)\, d\xi$$

$$\text{(FTC)} = \frac{1}{h}\left(\Pi(x_{\ell+\frac{1}{2}}) - \Pi(x_{\ell-\frac{1}{2}}) \right)$$

$$= \frac{1}{h}\left(V(x_{\ell+\frac{1}{2}}) - V(x_{\ell-\frac{1}{2}}) \right)$$

$$\text{(FTC)} = \frac{1}{h}\int_{x_{\ell-\frac{1}{2}}}^{x_{\ell+\frac{1}{2}}} u(\xi)\, d\xi = \bar{U}_\ell$$

Interpolation theory with equidistant nodes!

Via interpolation theory, we have

$$\Pi = V + O(h^{m+1}) \quad \text{over } [x_{j-r-\frac{1}{2}}, x_{j-r+m-\frac{1}{2}}]$$

can be computed just from cell averages via solution of a linear system of equations.

Hence

$$\Pi = u + O(h^m) \quad \text{over} \quad ''$$

Recall that there is an explicit formula for the polynomial interpolant, using Lagrange interpolants:

$$\Pi(x_{j-r+i-\frac{1}{2}}) = V(x_{j-r+i-\frac{1}{2}}), \quad i = 0, \dots, m$$

$$\Leftrightarrow \quad \Pi(x) = \sum_{i=0}^{m} V(x_{j-r+i-\frac{1}{2}})\, \ell_i^{(j-r-\frac{1}{2})}(x)$$

i-th Lagrange polynomial

One formula for the Lagrange polynomials is:

$$\ell_i^{(j-r-\frac{1}{2})}(x) = \prod_{\substack{0 \le t \le m \\ t \neq i}} \frac{x - x_{j-r+t-\frac{1}{2}}}{x_{j-r+i-\frac{1}{2}} - x_{j-r+t-\frac{1}{2}}}$$

degree m poly

We notice that $\ell_i^{(j-r-\frac{1}{2})}(x_{j-r+t-\frac{1}{2}}) = \delta_{it}$, and the partition of unity property

Kronecker delta

$$\sum_{i=0}^{m} \ell_i^{(j-r-\frac{1}{2})}(x) = 1$$

We consider

using partition of unity

$$\Pi(x) - V(x_{j-r-\frac{1}{2}}) = \sum_{i=0}^{m} \underbrace{\left[V(x_{j-r+i-\frac{1}{2}}) - V(x_{j-r-\frac{1}{2}}) \right]}_{h \sum_{q=0}^{i-1} \bar{U}_{q+j-r}} \ell_i^{(j-r-\frac{1}{2})}(x)$$

$$= \sum_{i=0}^{m} h \sum_{q=0}^{i-1} \bar{U}_{q+j-r} \, \ell_i^{(j-r-\frac{1}{2})}(x)$$

Hence

$$\pi(x) = \Pi'(x) = \sum_{i=0}^{m} h \sum_{q=0}^{i-1} \bar{U}_{q+j-r} \, \ell_i^{(j-r-\frac{1}{2})}(x)'$$

We rearrange the nested sum

$$\sum_{i=0}^{m} \sum_{q=0}^{i-1} a_q b_i = \sum_{q=0}^{m-1} \left(a_q \sum_{i=q+1}^{m} b_i \right)$$

$$\pi(x) = \sum_{q=0}^{m-1} \bar{U}_{q+j-r} \sum_{i=q+1}^{m} h \left(\ell_i^{j-r-\frac{1}{2}}\right)'(x)$$

We evaluate at the points $x_{j+\frac{1}{2}}$ and $x_{j-\frac{1}{2}}$

$$\pi(x_{j+\frac{1}{2}}) = \sum_{q=0}^{m-1} \bar{U}_{q+j-r} \underbrace{\sum_{i=q+1}^{m} h \left(\ell_i^{j-r-\frac{1}{2}}\right)'(x_{j-\frac{1}{2}})}_{C_{qr}^{m}}$$

$$= \sum_{q=0}^{m-1} \bar{U}_{q+j-r} \, C_{qr}^{m}$$

In summary, the polynomial satisfies the m (algebraic) average constraints and the approximation estimate.

constraints and the approximation estimate.

We can use the approximation

$$U_{j+\frac{1}{2}} = \sum_{i=0}^{m} C_{ir}^{m} \bar{U}_{j+i-r}$$

Remark: Above, the polynomial π is defined via m cell averages whereas Π is defined via $m+1$ cell interfaces.

How are we going to use this?

If r is fixed, we get a linear scheme.
Our idea is to choose r adaptively, that is, we adaptively shift the stencil. Generally speaking, we will try to avoid jumps.