We have seen that linear schemes only give approximations up to first-order. The same is true for monotone schemes.

Other schemes outside of the classes, such as the Lax-Wendroff scheme, exhibit oscillations.

Generally speaking, in search of higher order schemes we need to strike a delicate balance of quantitative approximation and qualitative stability.

Question: why are higher-order schemes of practical interest in the first place?

Example: Transport equation over $[0, 2\pi]$ with periodic BC

$$u_0(x) = \exp(i k x) \qquad \text{(single Fourier mode)}$$

Hence $u(x) = \exp(i k (x - ct))$ $\langle k = \text{wave number} \rangle$

Suppose we discretize the spatial derivative by a central difference quotient

$$\partial_t u_j(t) + c \frac{1}{2h}\left[u_{j+1}(t) - u_{j-1}(t) \right] = 0$$

More generally

$$\partial_t u(t) + c D_h u = 0$$

where

$$(D_h u)_j = \sum_{n=1}^{m} \alpha_n^m \frac{(E_n - E_{-n})}{2nh} u_j ,$$

$$E_n u_j = u_{j+n}$$

$$\alpha_n^m = -2 \cdot (-1)^n \frac{(m!)^2}{(m+n)!\,(m-n)!}$$

The central diff. quotient is the special case $m = 1$

The error in the derivative satisfies

$$\left| (D_h u)_j - \partial_x u(x_j) \right| \leq C h^{2m}$$

We conjecture that the solution to the semi-discrete scheme takes the form

$$v(x,t) = \exp\left(ik\left(x - c_{2m}(k)\, t\right)\right)$$

$c_{2m}(k) \rightarrow$ numerical phase speed for approx of order $2m$

We can then compute the num. phase speed by inserting v into the scheme.

For example,

$\boxed{m=1}$ $$\partial_t v = -c \, \frac{v(x+h) - v(x-h)}{2h}$$

$$\Rightarrow \quad c_2(k) = c \cdot \frac{\sin(kh)}{6\,kh} \approx c\left(1 - \frac{(kh)^2}{6} + \dots\right)$$

$\boxed{m=2}$

$$\Rightarrow \quad c_4(k) = c\, \frac{8\sin(kh) - \sin(2kh)}{6\,kh} \approx c\left(1 - \frac{(kh)^4}{30} + \dots\right)$$

We define the phase error

$$e_{2m}(k) := \left| \frac{u(x,t) - v(x,t)}{u(x,t)} \right|$$

$$= \left| 1 - \exp\left(ik\,(c - c_{2m}(k))\, t \right) \right|$$

$$\approx \left| k\,(c - c_{2m}(k))\, t \right|$$

Hence we get

$$e_2(k) = kct\, \frac{(kh)^2}{6}, \qquad e_4(k) = kct\, \frac{(kh)^4}{30}, \qquad \dots$$

We define

$$p = \frac{2\pi}{kh} = \frac{2\pi / h}{k} \qquad \text{points per wavelength}$$

$$\nu = \frac{kct}{2\pi} = k \cdot \frac{ct}{2\pi}$$ time periods to solve for

We write e_{2m} in terms of p and ν:

$$e_2(p,\nu) = \frac{\pi\nu}{3}\left(\frac{2\pi}{p}\right)^2$$

$$e_4(p,\nu) = \frac{\pi\nu}{15}\left(\frac{2\pi}{p}\right)^4$$

$$e_6(p,\nu) = \frac{\pi\nu}{70}\left(\frac{2\pi}{p}\right)^6$$

Question: Given an error threshold $\varepsilon > 0$, how large must p be for a given m and ν?

We write $p = p_m(\nu,\varepsilon)$ as a function of ν and ε.

$\varepsilon = 0.1$ $\quad p_2(\nu,\varepsilon) \geq 20\,\nu^{1/2}, \quad p_4(\nu,\varepsilon) \geq 7\,\nu^{1/4}, \quad p_6(\nu,\varepsilon) \geq 6\,\nu^{1/6}$

$\varepsilon = 0.01$ $\quad p_2(\nu,\varepsilon) \geq 64\,\nu^{1/2}, \quad p_4(\nu,\varepsilon) \geq 13\,\nu^{1/4}, \quad p_6(\nu,\varepsilon) \geq 8\,\nu^{1/6}$

$\varepsilon = 0.001$ $\quad \geq 643\,\nu^{1/2} \quad 43\,\nu^{1/4} \quad 26\,\nu^{1/6}$

Generally, the lower bound takes the form:

$$\boxed{p_{2m}(\nu,\varepsilon) \geq 2\pi \cdot g(m)\left(\frac{\nu}{\varepsilon}\right)^{1/m}}$$

Moral of the story: A higher order scheme requires less points when everything else is fixed.

We expect higher-order schemes to be helpful when ε is small while ν is large.

We remember linear schemes:

$$U_j^{n+1} = \sum c_\ell U_{j+\ell}^n$$

For second-order accuracy that is also MP we have

$$\left.\begin{aligned} &1)\quad \sum c_\ell = 1 \\ &2)\quad \sum \ell c_\ell = -\frac{ck}{h} \\ &3)\quad \sum \ell^2 c_\ell = \left(\frac{ck}{h}\right)^2 \end{aligned}\right\}$$ mutually exclusive

Consider the transport equation:

$$\partial_t u + c\,\partial_x u = 0 \qquad \lambda = \frac{ck}{h}$$

Lax-Wendroff:

$$U_j^{n+1} = U_j^n - \frac{\lambda}{2}\left(U_{j+1}^n - U_{j-1}^n\right) + \lambda^2\left(U_{j+1}^n - 2U_j^n + U_{j-1}^n\right)$$

What are the coefficients?

$$C_{-1} = \frac{\lambda(\lambda+1)}{2}, \qquad C_0 = 1-\lambda, \qquad C_1 = \frac{\lambda(\lambda-1)}{2}$$

If $|\lambda| \le 1$, then we have the CFL condition:

$$C_0 \ge 0, \qquad C_{-1}\cdot C_1 = \frac{1}{2}\lambda^2(\lambda-1)^2 \le 0$$

We then check:

$$1)\quad C_{-1} + C_0 + C_1 = 1$$

$$2)\quad -C_{-1} + 0\cdot C_0 + 1C_1 = -\lambda$$

$$3)\quad 1C_{-1} + 0\cdot C_0 + 1C_{-1} = \lambda^2$$

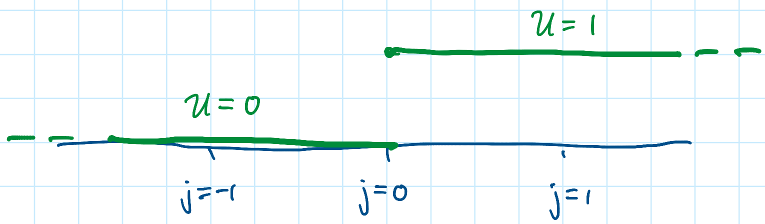
So we have second-order accuracy. But we don't have monotonicity

We consider the Riemann problem:

We consider the Riemann problem:

$u_j^0 = 0 \quad \text{if} \quad j < 0$

$u_j^0 = 1 \quad \text{if} \quad j \geq 0$



We study the Lax-Wendroff scheme in more detail

$j < -1 \qquad U_j^2 = 0$

$j = -1 \qquad U_j^1 = \lambda(\lambda - 1)/2$

$j = 0 \qquad U_j^1 = 1 - \frac{\lambda}{2} - \frac{\lambda^2}{2}$

$j = 1 \qquad U_j^1 = c_{-1} + c_0 + c_1 = 1$

$j > 1 \qquad U_j^1 = 1$

We have the CFL condition provided that $-1 \leq \lambda \leq 1$

If $\lambda > 0 \implies U_{-1}^1 < 0$ (Undershoot)

If $\lambda < 0 \implies U_0^1 > 1$ (Overshoot)

One can show such an effect for any linear second-order scheme.

We want to seek out non-oscillatory higher-order schemes.

Linear schemes are problematic:

$$\begin{tikzcd} \text{Monotone} \arrow[r, Rightarrow] & L^1 \arrow[r, Rightarrow] & \text{TVD} \arrow[r, Rightarrow] & \text{MP} \arrow[lll, bend left, Rightarrow, "\text{if linear}"] \end{tikzcd}$$

So we take a look at schemes that are non-linear.
It seems a reasonable bet to look for TVD schemes.