We begin the discussion with the linear system

$$\partial_t \underline{U} + \underline{\underline{A}}\, \partial_x \underline{U} = 0$$

matrix of size $m \times m$

$\underline{U} = (U_1, U_2, \dots, U_m)^T$ ← m functions in x and t

We call the system **hyperbolic** if the eigenvalues of $\underline{A}$ are real

We call the system **strongly hyperbolic** if A is diagonalizable, otherwise **weakly hyperbolic**

Moreover, we call it **strictly hyperbolic** if the eigenvalues are distinct.

If A is symmetric, then we call it **symmetric hyperbolic**.

For simplicity we assume that A is diagonalizable.

$$\underline{\underline{A}} = \underline{\underline{S}}\, \underline{\underline{\Lambda}}\, \underline{\underline{S}}^{-1}, \qquad \underline{\underline{S}} \in \mathbb{R}^{m\times m} \text{ invertible} \qquad \underline{\underline{\Lambda}} \text{ diagonal}$$

The diagonal entries of $\underline{\underline{\Lambda}}$ are the eigenvalues of $\underline{A}$, and the columns $\underline{S}_i$ are the eigenvectors.

We rewrite the conservation law, assuming A is constant,

$$\partial_t \underline{U} + \underline{\underline{A}}\, \partial_x \underline{U} = 0$$

$$\Rightarrow \quad \underline{\underline{S}}^{-1} \partial_t \underline{U} + \underline{\underline{S}}^{-1} \underline{\underline{A}}\, \underline{\underline{S}}\, \underline{\underline{S}}^{-1} \partial_x \underline{U} = 0$$

1. Multiply by $\underline{\underline{S}}^{-1}$
2. $\mathrm{Id} = \underline{\underline{S}}\, \underline{\underline{S}}^{-1}$

$$\Rightarrow \quad \partial_t \underline{V} + \underline{\underline{\Lambda}}\, \partial_x \underline{V} = 0$$

$\underline{V} = \underline{\underline{S}}^{-1} \underline{U}$

Those are m independent linear transport equations. The initial condition is

$$\underline{V}_0(x) = \underline{\underline{S}}^{-1} \cdot \underline{U}_0(x)$$

Hence we find for each component:

$$V_i(x,t) = V_{i,0}(x - \lambda_i t)$$

i-th eigenvalue of $\underline{\underline{A}}$

i-th diagonal entry of $\underline{\underline{\Lambda}}$

We can compute

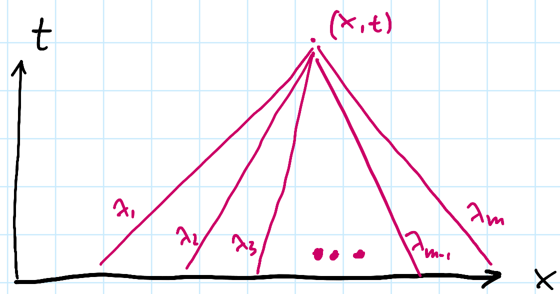
$$\underline{U}(x,t) = \underline{\underline{S}}\,\underline{V}(x,t) = \sum_{i=1}^{m} v_i(x,t)\,\underline{S}_i$$

$\underline{S}_i$: i-th eigenvector of $\underline{\underline{A}}$, i-th column of $\underline{\underline{S}}$

The exact solution is a sum of m waves propagating at the respective speeds λ_i. We can solve for each separately.

Domain of dependence

$\lambda_1 > \lambda_2 \ \dots \ > \lambda_{m-1} > \lambda_m$



<u>Example</u> Linear Wave equation

$$U_{tt} - U_{xx} = 0 \quad \Longleftrightarrow \quad \begin{aligned} U_t + V_x &= 0 \\ V_t + U_x &= 0 \end{aligned}$$

$$\begin{bmatrix} U \\ V \end{bmatrix}_t + \underbrace{\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}}_{=\underline{\underline{A}}} \begin{bmatrix} U \\ V \end{bmatrix}_x = 0$$

We have

$$\underline{\underline{A}} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \underline{\underline{S}}\,\underline{\underline{\Lambda}}\,\underline{\underline{S}}^{-1}$$

with

$$\underline{\underline{\Lambda}} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \qquad \underline{\underline{S}} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = \underline{\underline{S}}^{-1}$$

The solution is the linear combination of two waves propagating into opposite directions

$$\begin{bmatrix} U \\ V \end{bmatrix} = w_1(x,t) \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix} + w_2(x,t) \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix} \qquad \begin{aligned} w_1(x,t) &= w_{1,0}(x-t) \\ w_2(x,t) &= w_{2,0}(x+t) \end{aligned}$$