15 Finite Difference Schemes, revisited

We revisit the idea that the ratio of step sizes h/k should be small to ensure a stable scheme.

Consider the CL

$$u_t + c u_x = 0$$

$\boxed{\text{Example}}$ FTBS

$$U_j^{n+1} = U_j^n - \underbrace{\tfrac{k}{h}c}_{\lambda}\left(U_j^n - U_{j-1}^n\right) = (1-\lambda)U_j^n + \lambda U_{j-1}^n$$

Hence

$$|U_j^{n+1}| \le |U_j^n|\cdot|1-\lambda| + |U_{j-1}^n|\cdot|\lambda| \le \|U^n\|_\infty\left(|1-\lambda| + |\lambda|\right)$$

If $0 \le \lambda \le 1$, then

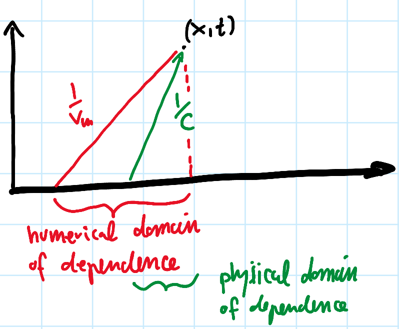
$$\|U^{n+1}\|_\infty \le \|U^n\|_\infty$$

Observe that

$$\lambda = \frac{k}{h}c = c\Big/\frac{h}{k}$$

Define mesh velocity $v_m = \frac{h}{k}$

Hence $0 \le \lambda \le 1$ requires $0 \le c$, $c \le v_m$



The physical domain of dependence must lie in the physical domain of dependence.

This is known as the CFL condition (Courant - Friedrich - Levy)

Example FTCS

$$U_j^{n+1} = U_j^n - \frac{1}{2} c \frac{k}{h}\left(U_{j+1}^n - U_{j-1}^n\right)$$

unstable :-(

A minor modification

$$U_j^{n+1} = \frac{U_{j+1}^n + U_{j-1}^n}{2} - \frac{1}{2} c \frac{k}{h}\left(U_{j+1}^n - U_{j-1}^n\right)$$

$$= U_j^n - \frac{k}{h}\left(F_{j+\frac{1}{2}}^n - F_{j-\frac{1}{2}}^n\right)$$

where

$$F_{j+\frac{1}{2}}^n = \frac{c}{2}\left(U_{j+1}^n + U_j^n\right) - \frac{h}{2k}\left(U_{j+1}^n - U_j^n\right)$$

stable :-)

Exercise: this is a monotone scheme if $|\lambda| \le 1$, $\lambda = c/v_m$

Review $U_j^{n+1} = U_j^n - \frac{k}{h}\left(F_{j+\frac{1}{2}}^n - F_{j-\frac{1}{2}}^n\right)$

1) $$F(u,v) = \frac{f(u) + f(v)}{2} - \frac{\alpha}{2}(v-u)$$ $\alpha = \max f' \le \frac{k}{h}$

Lax-Friedrich

2) $$F(u,v) = \frac{f(u) + f(v)}{2} - \frac{\tilde{\alpha}}{2}(v-u)$$

$\tilde{\alpha} = \max\{ f'(u), f'(v) \}$ "Local Lax-Friedrich" / Rusanov

3) $$F_{LW}(u,v) = \frac{f(u) + f(v)}{2} - \frac{k}{2h} f'\left(\frac{u+v}{2}\right)\left(f(v) - f(u)\right)$$

Lax-Wendroff

Example: Application to $u_t + c u_x = 0$

LF, Rus $\rightarrow$ smooth, LW $\rightarrow$ oscillation

We find a smooth $v(x,t)$ satisfying the LF scheme exactly at the grid points.

$$v(x_j, t^{n+1}) = \frac{1}{2}\left(v(x_{j+1}, t^n) + v(x_{j-1}, t^n)\right) - \frac{kc}{2h}\left(v(x_{j+1}, t^n) - v(x_{j-1}, t^n)\right)$$

Taylor expansion:

$$v_t + c v_x = \frac{k}{2}\left(\frac{h^2}{k^2} v_{xx} - v_{tt} \right) + O(h^2, k^2)$$

we then take derivatives in x and t and find

$$v_{tt} = c v_{xx} + O(k)$$

$$\lambda = c\frac{h}{k}$$

Modified PDE of a numerical scheme

$$v_t + c v_x = \frac{h^2}{2k}(1 - \lambda^2) v_{xx} + O(k^2)$$

$\boxed{LW}$

$$v_t + c v_x = -\frac{h^2}{6}(1 - \lambda^2) v_{xxx}$$

dispersive eq.
waves travel at diff. speeds

Example (Roe scheme)

We recall the Roe scheme

$$F(u,v) = \begin{cases} f(u) & \text{if } s \geq 0 \\ f(v) & \text{if } s \leq 0 \end{cases} \qquad s = \frac{f(u) - f(v)}{u - v}$$

For Burgers' equation, shock instead of rarefaction when

$$u_l \leq u_s \leq u_r \qquad u_l = -1, \quad u_r = 1$$

We modify the scheme

$$F(u,v) = \begin{cases} \max\limits_{z \in [v,u]} f(z) & u > v \\ \min\limits_{z \in [u,v]} f(z) & u < v \end{cases}$$