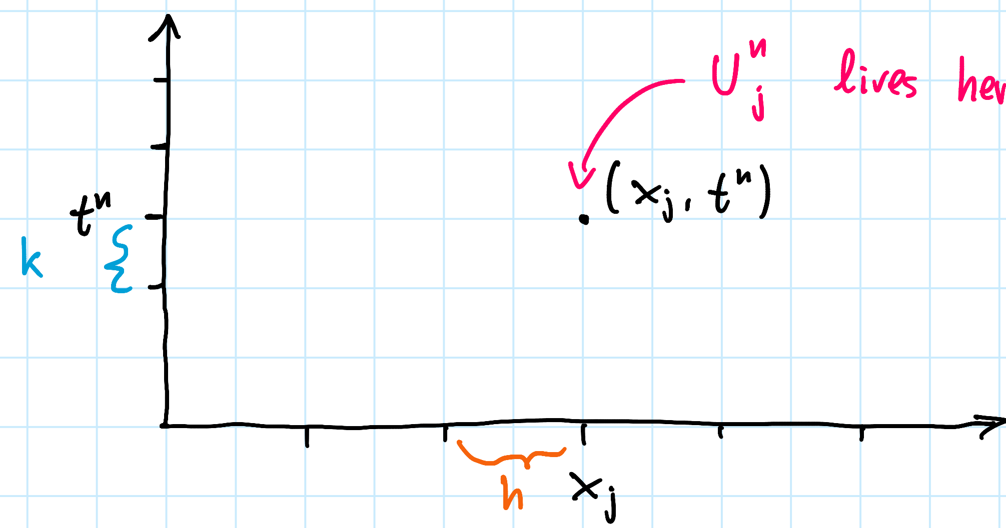
We introduce a few basic numerical schemes for the conservation law

$$\partial_t u + \partial_x f(u) = 0$$

We work over a finite space-time domain $\Omega_x \times \Omega_t$

$$\Omega_x = [A, B], \qquad \Omega_t = [0, T]$$

We introduce nodal points over Ω_x of equal distance h.
We introduce time steps over Ω_t of equal distance k.



Specifically:

$$x_j = A + j \cdot h, \qquad j = 0, \dots, N \qquad h = \frac{B-A}{N}$$

$$t^n = n \cdot k, \qquad n = 0, \dots, K \qquad k = T/K$$

We have discretized space and time. Now we discretize the unknown function u. We introduce

$$U_j^n \qquad \text{associated to} \qquad (x_j, t^n)$$

Given initial data and boundary conditions at the nodes, we want to find values U_j^n such that

$$U_j^n \approx u(x_j, t^n)$$

How do we define U_j^n then?

Recall finite differences: we approximate the derivatives in the conservation law by difference quotients.

Forward Difference:

$$\partial_x u(x_j, t^n) \approx \frac{U_{j+1}^n - U_j^n}{h} + \mathcal{O}(h)$$

Backward Difference

$$\partial_x u(x_j, t^n) \approx \frac{U_j^n - U_{j-1}^n}{h} + \mathcal{O}(h)$$

Central Difference

$$\partial_x u(x_j, t^n) \approx \frac{U_{j+1}^n - U_{j-1}^n}{2h} + \mathcal{O}(h^2)$$

Similarly

$$\partial_t u(x_j, t^n) \approx \frac{U_j^{n+1} - U_j^n}{k}$$

$$\partial_t u(x_j, t^n) \approx \frac{U_j^{n} - U_j^{n-1}}{k}$$

$$\left(\partial_t u(x_j, t^n) \approx \frac{U_j^{n+1} - U_j^{n-1}}{2k} \right)$$

We replace the actual derivatives by finite differences and isolate variables so that we can compute the nodal values

Let's approach the transport equation

$$\partial_t u + c\,\partial_x u = 0,$$

E.g. Forward-time backward space

$$\partial_t u + c\,\partial_x u = 0$$

$$\Leftrightarrow \quad \frac{U_j^{n+1} - U_j^n}{k} + c\,\frac{U_j^n - U_{j-1}^n}{h} = 0$$

We isolate U_j^{n+1} to find

$$U_j^{n+1} = U_j^n - c\,\frac{k}{h}\left(U_j^n - U_{j-1}^n\right)$$

Now, given initial values at $t=0$ $(n=0)$ and boundary conditions, we can compute U_j^n going from time step to time step.

boundary conditions, we can compute U_j going from time step to time step.

E.g. Forward time forward space

$$\partial_t u + c \partial_x u = 0$$

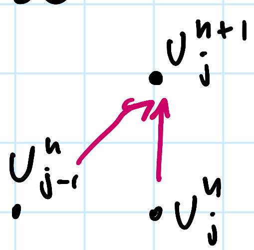
$$\Leftrightarrow \quad \frac{U_j^{n+1} - U_j^n}{k} + c \, \frac{U_{j+1}^n - U_j^n}{h} = 0$$

We isolate U_j^{n+1}

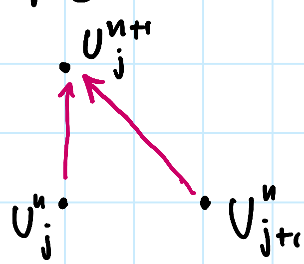
$$U_j^{n+1} = U_j^n - c \frac{k}{h} \left(U_{j+1}^n - U_j^n \right)$$

Graphically:

FTBS



FTFS



What is a good method $c = 1$, that is

$$\partial_t u + \partial_x u = 0 \quad ?$$

We have seen a few formulas to calculate a time step from the next one.

FTBS $$U_j^{n+1} = U_j^n - c \frac{k}{h} \left(U_j^n - U_{j-1}^n \right)$$

FTFS $$U_j^{n+1} = U_j^n - c\frac{k}{h}\left(U_{j+1}^n - U_j^n\right)$$

FTCS $$U_j^{n+1} = U_j^n - \frac{ck}{2h}\left(U_{j+1}^n - U_{j-1}^n\right)$$