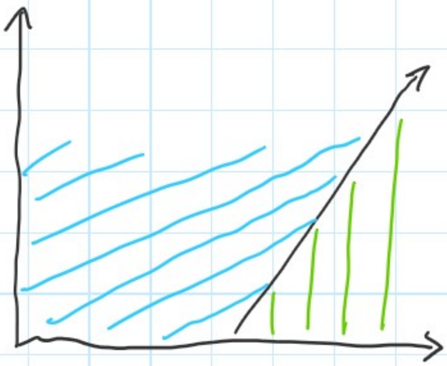
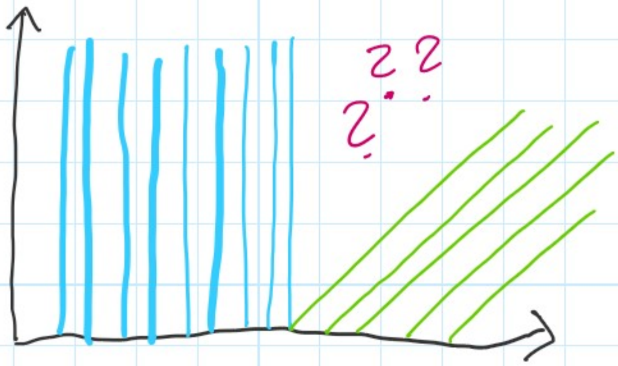
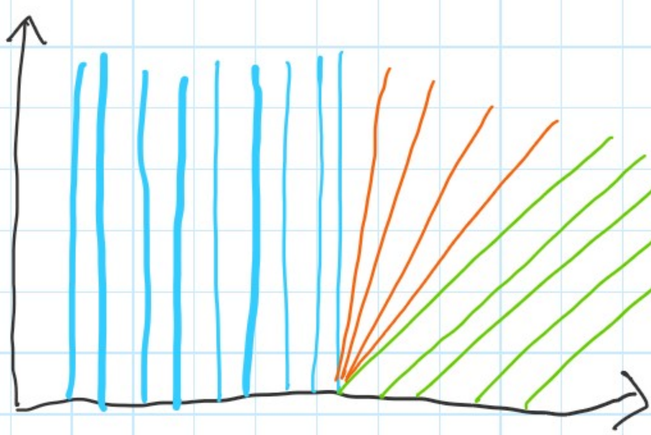
We have seen that weak solutions for Burgers equation generally are not unique.

This happens when characteristics don't cover the entire space-time region



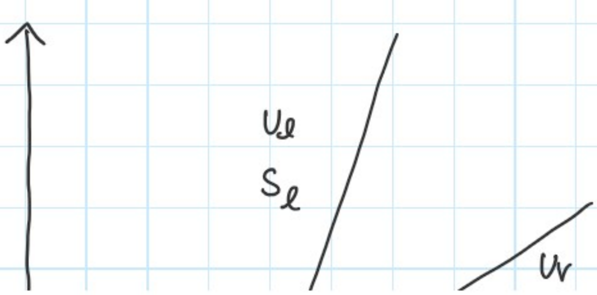


It seems reasonable that the characteristics emerge from the discontinuity, as to ensure a causal connection with the initial data.



How do we calculate the slope and values of the new characteristics?

The slope is already given, we use Rankine-Hugoniot to find the values.



To the left, the RH condition says

$$f(u_l) - f(u) = s\,(u_l - u)$$

To the right, the RH condition says

$$f(u_r) - f(u) = s(u_r - u)$$

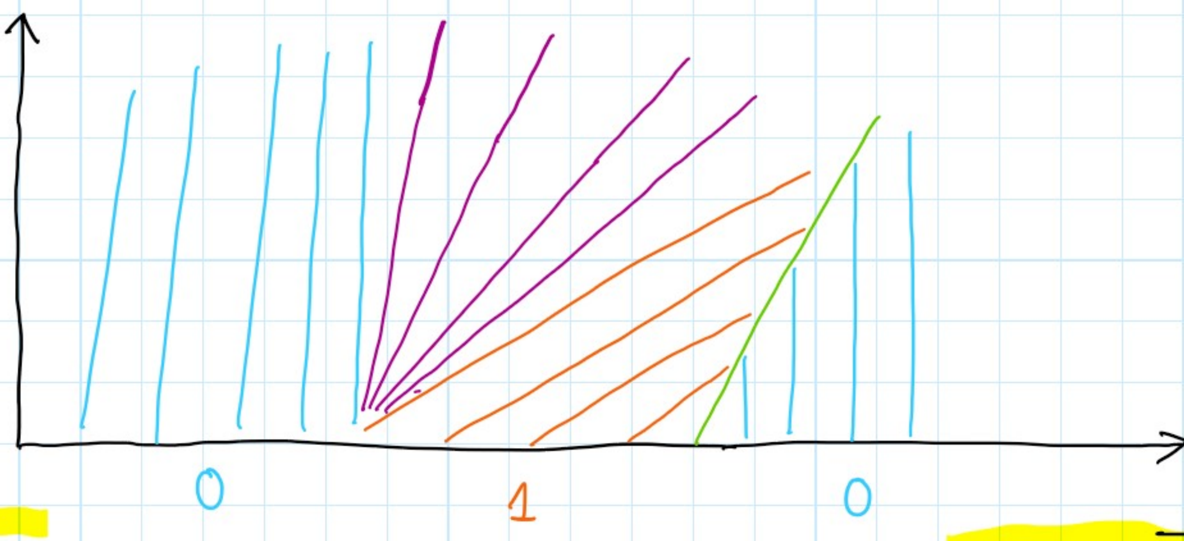
The slope is largest to the left and smallest to the right, so

$$\underbrace{\frac{f(u_\ell) - f(u)}{u_\ell - u}}_{\text{slope to the left}} \geq s \geq \underbrace{\frac{f(u_r) - f(u)}{u_r - u}}_{\text{slope to the right}}$$

u inbetween u_ℓ and u_r

This is called $\boxed{\text{Lax Entropy condition}}$

Intuition about time ...

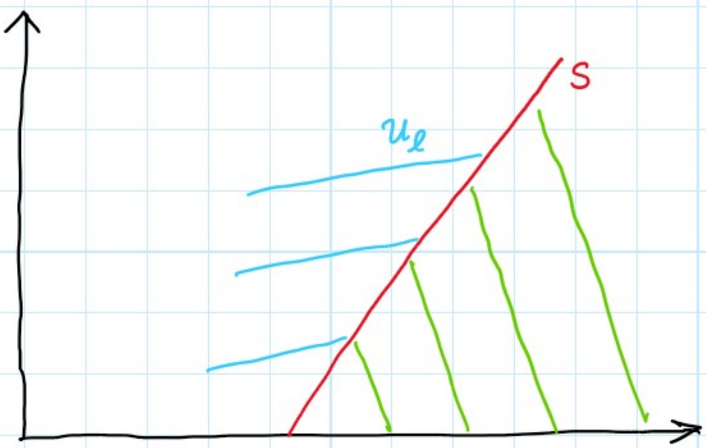


Note that characteristics run into shocks but do not emerge from it. The Lax entropy condition can be interpreted in that way.

When f is differentiable, then no characteristics

When f is differentiable, then no characteristics emerging from a discontinuity with speed s means

$$f'(u_\ell) \geq s \geq f'(u_r)$$



Are weak solution satisfying the entropy condition, called (weak) entropy solutions, unique?

<u>Theorem</u> Let $f: \mathbb{R} \to \mathbb{R}$ be strictly convex. Suppose that u and v are weak solutions to

$$\partial_t u + \partial_x f(u) = 0, \quad u(x,0) = u_0(x)$$
$$\partial_t v + \partial_x f(v) = 0, \quad v(x,0) = v_0(x)$$

such that u and v are piecewise smooth and along all discontinuities they satisfy the entropy condition.

Assume that $u, v \in L^1(\mathbb{R})$. Then

$$\partial_t \| u(t) - v(t) \|_{L^1(\mathbb{R})} \leq 0.$$

Proof: Hesthaven, Theorem 2.3

Generally speaking, no matter the initial data, the entropy solutions converge to each other in L^1 norm

Corollary: Let $f: \mathbb{R} \to \mathbb{R}$ be strictly convex. Let u be a piecewise smooth entropy solution to

$$\partial_t u + \partial_x f(u) = 0, \quad u(x,0) = u_0(x)$$

with initial data $u_0 \in L^1(\mathbb{R})$.

Then u is unique.

Proof: We apply the preceding theorem. If u, v are two different solutions to

$$\partial_t u + \partial_x f(u) = 0, \qquad u(x,0) = u_0(x)$$
$$\partial_t v + \partial_x f(v) = 0, \qquad v(x,0) = u_0(x)$$

By the preceding theorem and find

$$\partial_t \| u(t) - v(t) \| \leq 0 \quad \Longrightarrow$$ The L^1-norm of $u-v$ does not increase over time

and also at $t=0$

$$\| u(0) - v(0) \|_{L^1} = 0$$

So $\| u(t) - v(t) \|_{L^1} = 0$. Hence $u = v$ (almost everywhere) at all times. $\square$

Having shown uniqueness of entropy solutions, we proceed with showing their existence

We consider the modified conservation law

Remark: Heat equation
$\partial_t u = \partial_{xx}^2 u$

We consider the modified conservation law

heat equation
$$\partial_t u = \partial^2_{xx} u$$

$$\partial_t u_\varepsilon + \partial_x f(u_\varepsilon) = \varepsilon\, \partial^2_{xx} u_\varepsilon$$

with additional viscosity term. One can show that the solution exists in L^1 and converges to a weak solution u

$$u_\varepsilon \longrightarrow u \quad \text{in } L^1 \quad \text{as} \quad \varepsilon \to 0$$

that satisfies the entropy condition.

Example Assume that f is strictly convex,

$\Rightarrow f'$ strictly increases $\Rightarrow f'' > 0$

What can we tell about conservation laws where the flux is strictly convex?

Let u be a weak solution to the conservation law

$$\partial_t u + \partial_x (f(u)) = 0$$

with a discontinuity along a curve with speed s. Suppose that u is an entropy solution.

$\boxed{\text{We can't have } u_\ell < u_r}$

Let us assume $u_\ell < u_r$ and show a contradiction.

Using the mean value theorem, for $u \in (u_\ell, u_r)$

$$s_\ell = \frac{f(u) - f(u_\ell)}{u - u_\ell} = f'(\xi_1) \qquad \xi_1 \in (u_\ell, u)$$

$$s_r = \frac{f(u) - f(u_r)}{} = f'(\xi) \qquad \xi \in (u, u_r)$$

$$S_r = \frac{f(u) - f(u_r)}{u - u_r} = f'(\xi_2) \qquad \xi_2 \in (u, u_r)$$

Since f strictly increases, $f'(\xi_1) < f'(\xi_2)$, thus

$$S_\ell < S_r$$

which violates the entropy condition. $\lightning$

So we can't have $u_\ell < u_r$.