04 Weak solutions and RH condition

Consider a conservation law

$$\partial_t u + \partial_x f(u) = 0, \qquad u(0,x) = u_0(x)$$

We multiply that with a test function ϕ, that is, a smooth function with compact support, and then we take integrals. We find that

$$\iint \partial_t u \cdot \varphi + \partial_x f(u) \cdot \varphi \, dx\,dt = 0$$

Both equations are equivalent for a smooth solution u.
Now we integrate by parts

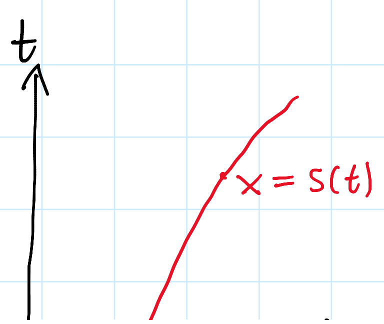
$$\iint u \, \partial_t \varphi + f(u) \, \partial_x \varphi \, dt\,dx + \int u_0 \nu \, dx = 0$$

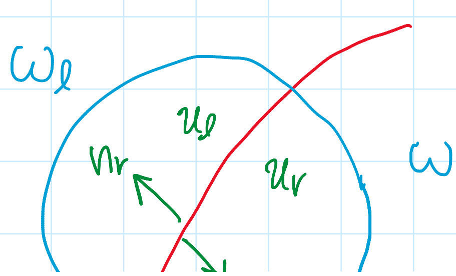
This is the weak formulation of the initial value problem. If any locally integrable function u satisfies the weak formulation for all test functions ϕ, then we call it a weak solution.

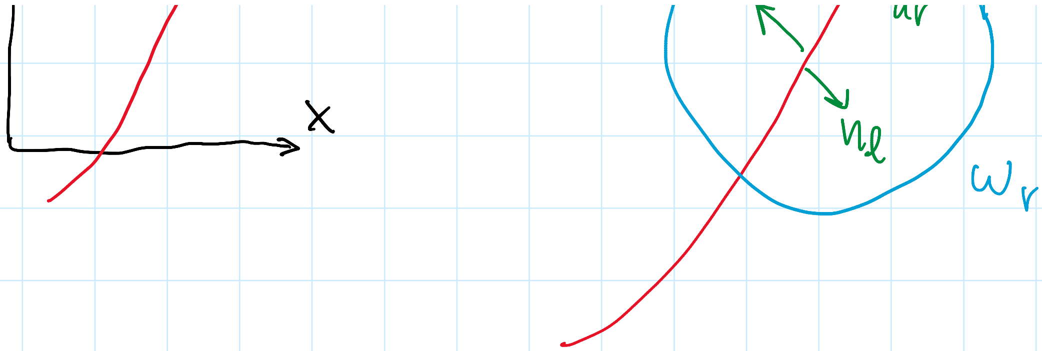
Rankine-Hugoniot condition

Suppose that u is piecewise differentiable with a discontinuity along a curve $x = s(t)$ and that u is a weak solution to the conservation law.

$$\partial_t u + \partial_x f(u) = 0$$







Consider a small domain ωthat covers a part of the curve. For any test function ϕ supported within the small domain ω, we observe

$$0 = \iint_{\omega} u\,\varphi_t + f(u)\,\varphi_x \, dx\,dt$$

$$= \iint_{\omega_l} u\,\varphi_t + f(u)\,\varphi_x \, dx\,dt + \iint_{\omega_r} u\,\varphi_t + f(u)\,\varphi_x \, dx\,dt$$

$$= \iint_{\omega_l} u_t\,\varphi + f(u)_x\,\varphi \, dx\,dt + \iint_{\omega_r} u_t\,\varphi + f(u)_x\,\varphi \, dx\,dt$$

$$+ \oint_{\omega_l} \left(u_l\,\vec{n}^{\,l}_t + f(u_l)\,\vec{n}^{\,l}_x\right)\varphi + \oint_{\omega_r} \left(u_r\,\vec{n}^{\,r}_t + f(v_r)\,\vec{n}^{\,r}_x\right)\varphi$$

Only the boundary integrals remain. Since u is continuous on both sides we see

$$(u_l - u_r)\,\vec{n}^{\,l}_t + \left(f(u_l) - f(u_r)\right)\vec{n}^{\,l}_x = 0$$

$$\Rightarrow \quad (u_\ell - u_r)(-\dot{s}(t)) + f(u_\ell) - f(u_r) = 0$$

$$\Rightarrow \quad f(u_\ell) - f(u_r) = \dot{s}(t)(u_\ell - u_r)$$

speed of propagation
of discontinuity