Exercise Set 8: Gudonov's Method

Exercise 1
Discuss qualitatively the derivation of *Godunov's method*; sketch each step in the solution process. Which part of the algorithm can make its implementation particularly difficult?

Exercise 2
Consider the scalar conservation law

$$u_t + f(u)_x = 0 \ , \tag{1}$$

and initial condition

$$u(x,0) = \begin{cases} u_l & x < 0 \\ u_r & 0 < x \end{cases} \ , \tag{2}$$

where the flux f is convex ($f'' > 0$). Godunov's method relies on finding the intermediate state $u^* = u^*(u_l, u_r)$ for which $u(0,t) = u^*$, for $t > 0$.

1. Show that this intermediate state is given by the following:

   1. $f'(u_l), f'(u_r) \geq 0 \implies u^* = u_l$
   2. $f'(u_l), f'(u_r) \leq 0 \implies u^* = u_r$
   3. $f'(u_l) \geq 0 \geq f'(u_r) \implies u^* = \begin{cases} u_l & s > 0 \\ u_r & s < 0 \end{cases} \ , \qquad s = \dfrac{f(u_r) - f(u_r)}{u_r - u_l}$
   4. $f'(u_l) < 0 < f'(u_r) \implies u^* = u_m \ ,$ where u_m is the solution to $f'(u_m) = 0$.

2. Use (a) to show that Godunov's flux is given by

$$F(u_l, u_r) = \begin{cases} \min\limits_{u_l \leq u \leq u_r} f(u) & u_l \leq u_r \\ \max\limits_{u_r \leq u \leq u_l} f(u) & u_l > u_r \end{cases} \ . \tag{3}$$

3. Show that Godunov's flux (3) is monotone.

Exercise 3
The purpose of this exercise is to illustrate the *Lax-Wendroff Theorem*. This theorem states that if there exists a sequence $\{(h_l, k_l)\}_l^\infty$ with $k_l = \lambda h_l$ (λ is kept constant), such that the corresponding numerical solutions $\{v_l\}_{l=1}^\infty$ obtained by a conservative method converges to some function u, then the limit u is a weak solution of the conservation law. Notice that to deduce the conclusion, we assume that v_l converges as $l \to \infty$. That is, convergence is not a conclusion of the Lax-Wendroff theorem. Also recall that in general weak solutions are not unique, so the theorem does not guarantee the limit is the correct entropy solution.

Consider a conservative method

$$v_j^{n+1} = v_j^n - \frac{k}{h}\left(F\left(v_j^n, v_{j+1}^n\right) - F\left(v_{j-1}^n, v_j^n\right)\right) \tag{4}$$

where the numerical flux F is given by

$$F(v,w) = \begin{cases} f(v) & \frac{f(v)-f(w)}{v-w} \geq 0 \\ f(w) & \frac{f(v)-f(w)}{v-w} < 0 \end{cases} \ . \tag{5}$$

1. Construct the entropy solution to the following initial value problem

$$u_t + \left(\frac{1}{2}u^2\right)_x = 0 \quad u(x,0) = \begin{cases} -1 & x < 1 \\ 1 & x > 1 \end{cases}. \tag{6}$$

2. Fix $k/h = 0.5$, and implement the above method to (6), in $x \in (0,2)$, $0 < t \leq 0.25$, with the initial data discretized using cell averages. On the boundaries, set $u(0,t) = -1$, and $u(2,t) = 1$.

3. Run the computations by choosing i) $h_l = \frac{2}{l}$, ii)$h_l = \frac{2}{2l}$, iii) $h_l = \frac{2}{2l+1}$, for $l \in \mathbb{N}$.

4. What can you deduce from your results regarding the each of the three sequences of numerical solutions obtained. Explain your results and conclude how they fit with the Lax-Wendroff theorem.